The Korean Journal for the History of Science 34-2 (2012), 287–357.

Tracing Practices Purloined by the "Three Pillars"*

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This article traces the history of a series of problems included in the *Guide to Calculation in the Unified Script* (*Tong wen suan zhi* 同文算指, 1613), attributed to Matteo Ricci (1552–1610) and Li Zhizao 李之藻 (1565–1630).[1] The *Guide to Calculation* was purported to be a translation of Christoph Clavius's (1538–1612) *Epitome arithmeticae practicae* (1583),[2] together with some problems from Chinese mathematical treatises that were included for comparison to demonstrate the alleged superiority of Western mathematics. Li Zhizao, Xu Guangqi 徐光啟 (1562–1633), and Yang Tingyun 杨廷筠 (1557–1627), who collectively are often referred to as the "Three Pillars" (*san da zhushi* 三大柱石) of Catholicism in Ming China, each wrote a preface to the *Guide to Calculation* denouncing contemporary Chinese mathematics while promoting the superiority of Western mathematics. My efforts here will focus on Xu's preface, in which he argues that Chinese mathematics had been in a state of decline ever since the Zhou Dynasty (1045?–256 BCE), and all that remained of it was vulgar and corrupt. Western mathematics was in every way superior, he asserted, and

* Received 23 June 2012; revised 26 July 2012; accepted 20 August 2012. I thank Professors Yung Sik Kim 金永植, Jongtae Lim 林宗台, and others at Seoul National University for their patient guidance and generous assistance. In particular I thank Professor Lim for generously offering detailed written corrections, criticisms, suggestions, and improvements. I also thank John N. Crossley, A. G. Hopkins, and Alberto Martinez for their suggestions and encouragement. Portions of this paper are from *Imagined Civilizations: China, the West, and Their First Encounter* (Baltimore: Johns Hopkins University Press, in press, forthcoming in 2013), *The Chinese Roots of Linear Algebra* (Baltimore: Johns Hopkins University Press, 2011), and two articles in preparation, "The Diffusion of Linear Algebra across Medieval Eurasia" and "Reconstructing Mathematical Practices from Fibonacci's *Liber Abaci*: 'On the Buying of Horses.'"

† Much of this research was completed under a Visiting Fellowship from the Templeton Science and Religion in East Asia Project, Science Culture Research Center, Seoul National University, under the sponsorship of the Templeton Religion Trust. I am most grateful for this support.

1 *Tong wen suan zhi* 同文算指 [Guide to calculation in the unified script], in *Zhongguo kexue jishu dianji tonghui: Shuxue juan* 中國科學技術典籍通彙：數學卷 [Comprehensive collection of the classics of Chinese science and technology: Mathematics volumes] (Zhengzhou: Henan jiaoyu chubanshe 河南教育出版社, 1993), hereinafter *ZKJDT*. For an analysis of the *Guide to Calculation*, see Takeda Kusuo 武田楠雄, *Dōbunzanshi no seiritsu* 同文算指の成立 [Inception of the Guide to calculation in the unified script] (Tokyo: Iwanami Shoten 岩波書店, 1954). For a recent study of the transmission of Western Learning into China, see Ahn Daeok 安大玉, *Minmatsu Seiyō kagaku tōdenshi: "Tengaku shokan" kihen no kenkyū* 明末西洋科学東伝史:「天学初函」器編の研究 [History of the transmission of Western science to the East in the late Ming: Research on the Qibian of the Tianxue chuhan] (Tokyo: Chisen Shokan 知泉書館, 2007).

2 Christoph Clavius, SJ (1538–1612), *Christophori Clavii Bambergensis e Societate Iesv epitome arithmetica practicae* (Romae: Ex Typographia Dominici Basae, 1583).

in the end, he alleged, the loss of Chinese mathematics was no more to be regretted than "discarding tattered sandals" (*qi bi jue* 棄敝屩).

The problems purloined by the Three Pillars and their collaborators, problems that were known in imperial China as *fangcheng* 方程 (sometimes translated as matrices or "rectangular arrays"), are arguably the most advanced and recognizably "modern" mathematics in the *Guide to Calculation*. These problems were copied into chapter 5 of the second volume (*tong bian* 通編), without attribution or any indication that they were from contemporary Chinese treatises. These problems were then given a new name, the title of the chapter, "Method for addition, subtraction, and multiplication of heterogeneous [elements]" (*za he jiao cheng fa* 雜和較乘法). Chinese readers of the *Guide to Calculation* could not have known that Clavius's *Epitome* contains no similar problems, but later Chinese commentators added notes remarking that similar problems could be found in Chinese works.

That nineteen *fangcheng* problems were purloined from the very Chinese mathematical texts denounced by Xu as "tattered sandals" suggests that Li, Xu, Yang, and their collaborators did not themselves believe their assertions of the superiority of mathematics from "the West." Yet Xu's pronouncements have seemed so persuasive that his claims have been, at least until recently, accepted for the most part by historians—Chinese and Western alike—as fact. We should instead critically analyze the self-serving statements in their prefaces as propaganda designed to promote "Western Learning" (*Xi xue* 西學).

A second purpose of this article is to question the assumption that the arrival of the Jesuits in Ming China marks the introduction of Western science into China and the "first encounter" of China and the West. This article traces the history of these *fangcheng* problems to show that we reach very different conclusions if we study mathematical practices themselves, instead of just focusing on the texts that preserve written records of these practices. In particular, tracing the history of *fangcheng* practices leads to the following conclusions: (1) The essentials of the methods used today in "Western" linear algebra—augmented matrices, elimination, and determinantal-style calculations—were known by the first century CE in imperial China. (2) Simple two-dimensional patterns were used to calculate determinantal-style solutions to a special class of distinctive problems. (3) These practices were non-scholarly—they did not require literacy and were not transmitted by texts. (4) These practices spread across the Eurasian continent and are recorded in texts in Italy from as early as the thirteenth century. These practices, in other words, were not confined by the boundaries we anachronistically term "civilizations." In particular, these practices circulated across Eurasia long before the Jesuits traveled in the late sixteenth and early seventeenth centuries to China.

These conclusions can help us begin to rewrite the history of the Jesuits in China as part of a long history of global circulations (as opposed to a "first encounter" of two civilizations, China and the West). It can also help us rethink one of the fundamental assumptions of the history of science—an assumption found in conventional histories of (Western) science, in Needham's *Science and Civilisation in China*, and in more recent comparative and cultural histories—that science in the premodern period was somehow "local" and transmitted by texts. The broader goal of this article then is to propose an alternative approach to the history of science, one that focuses on practices instead of texts, microhistory instead of macrohistory, and goes beyond "civilizations" toward a world history of science.

This article is divided into six sections. The first section uses extant written records to reconstruct *fangcheng* as a mathematical practice on the counting board, summarizing research from my recent book on *fangcheng*,[3] in order to demonstrate how adepts of this practice could quickly solve complex mathematical problems with n conditions in n unknowns with nothing more than facility with counting rods and the rote application of simple patterns on the counting board. The second section inquires into the provenance of extant written records of *fangcheng* practices. Written records of *fangcheng* practices are preserved in treatises on the mathematical arts, which were compiled by aspiring literati and presented to the imperial court as essential to ordering the empire. These literati understood only the rudiments of these practices, yet they derided methods used by adepts as arcane. The third section presents a summary of determinantal-style calculations and solutions to *fangcheng* problems, solutions seemingly so arcane that they were only rarely recorded by the literati who compiled mathematical treatises. The fourth section presents evidence that these arcane determinantal-style calculations and solutions—which are so distinctive that they can serve as "fingerprints"—circulated across the Eurasian continent, to be recorded in the works of Leonardo Pisano (c. 1170–c. 1250), more commonly known today by the name Fibonacci. The fifth section analyzes how *fangcheng* problems were purloined by Li, Xu, Yang, and their collaborators to support their claim that Western mathematics was in every way superior to Chinese mathematics. The concluding section outlines an alternative approach to the world history of science.

The *Fangcheng* Procedure as a Mathematical Practice

Today, linear algebra is one of the core courses in modern mathematics, and its main problem is the solution of systems of n linear equations in n unknowns.[4] Determinants, through what is known as Cramer's rule, offer an elegant solution for simple systems of linear equations; Gaussian elimination is the more general solution, and arguably the most important of all matrix algorithms.[5] The earliest extant records of these two approaches to solving linear equations can be found in extant mathematical treatises from imperial China.[6] This section will focus on the earliest records of the procedure we now call Gaussian elimination (determinantal calculations and solutions will be examined in the third section of this article).

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[3] Roger Hart, *The Chinese Roots of Linear Algebra* (Baltimore: Johns Hopkins University Press, 2011).

[4] Gilbert Strang, *Linear Algebra and Its Applications*, 4th ed. (Belmont, CA: Thomson, Brooks/Cole, 2006).

[5] G. W. Stewart, *Matrix Algorithms* (Philadelphia: Society for Industrial and Applied Mathematics, 1998); Strang, *Linear Algebra and Its Applications*.

[6] Hart, *Chinese Roots of Linear Algebra*.

The earliest known records of what we now recognize as Gaussian elimination are found in the *Nine Chapters on the Mathematical Arts* [*Jiuzhang suanshu* 九章算術],[7] in the eighth of those chapters, which is titled "*Fangcheng*" 方程. In that chapter, 18 problems are recorded as word problems, without diagrams. These problems range from two conditions in two unknowns to five conditions in five unknowns; there is also one problem, problem 13, with five conditions in six unknowns, which I will refer to as the "well problem."[8] The *Nine Chapters* stipulates that all 18 problems are to be solved by the *fangcheng* procedure (*fangcheng shu* 方程術), a variant of which we now call Gaussian elimination.[9] This section will illustrate how the *fangcheng* procedure, as reconstructed from written records preserved in the *Nine Chapters*, was a mathematical practice that used simple patterns on a counting board to produce solutions to what we would now call systems of linear equations with *n* conditions in *n* unknowns, in a manner similar to what we now call Gaussian elimination.

First, however, we must distinguish between written records of the *fangcheng* procedure preserved in extant Chinese mathematical treatises and the *fangcheng* procedure itself as a mathematical practice. As a mathematical practice, the *fangcheng* procedure was performed on a two-dimensional counting board with counting rods. Numbers were recorded using differing arrangements of counting rods. The Chinese numeral-rod system is not difficult to understand: the first position is for numbers 0 to 9 (0 is denoted by an empty space); 1 to 5 are denoted by the corresponding numbers of rods placed vertically, and a horizontal rod denoting 5 is placed above one to four vertical rods to denote 6 to 9. The next position to the left is for tens, which are placed horizontally. Then hundreds, and so on. A summary of the representation of numbers by counting rods is shown below in Table 1.

Table 1: Chinese rod numerals for one through nine, and ten through ninety; zero is denoted by a blank space, as indicated here for numbers 10 through 90.

1	2	3	4	5	6	7	8	9
𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
10	20	30	40	50	60	70	80	90
𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

7 *Jiuzhang suanshu* 九章算術 [Nine chapters on the mathematical arts], in *Yingyin Wenyuan ge Siku quanshu* 影印文淵閣四庫全書 [Complete collection of the four treasuries, photolithographic reproduction of the edition preserved at the Pavilion of Literary Erudition] (Hong Kong: Chinese University Press, 1983–1986), hereinafter *SKQS*.

8 I thank Jean-Claude Martzloff for suggesting this name. A version of the well problem is presented below, on pages 345ff. See also Hart, *Chinese Roots of Linear Algebra*, chapter 7, "The Well Problem."

9 Shen Kangshen, Anthony W.-C. Lun, and John N. Crossley, *The Nine Chapters on the Mathematical Art: Companion and Commentary* (New York: Oxford University Press, 1999); Karine Chemla and Guo Shuchun, *Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires* (Paris: Dunod, 2004); Hart, *Chinese Roots of Linear Algebra*.

Thus, for example, the number 721 is arranged on the counting board as 𝍦 𝍪 𝍠 .

The counting board was a powerful tool for the computation of solutions to mathematical problems ranging from addition, subtraction, multiplication, and division to the extraction of roots and the solution of systems of linear equations, namely, *n* conditions in *n* unknowns.[10] Although there are no modern practitioners, it seems reasonable to assume that counting rod adepts in imperial China had skills similar to those of modern abacus adepts. With practice, abacus calculations are fast, efficient, and virtually effortless, utilizing what is sometimes called motor learning or procedural memory, in which repetitive practice results in long-term muscle memory allowing calculations to be performed with little conscious effort.[11] Modern abacus adepts can calculate very quickly; some become so proficient that they use only procedural memory and do not require a physical abacus. It seems reasonable to assume that counting rods were similarly fast, efficient, and virtually effortless for accomplished adepts.

Expertise on a counting board, as with the abacus, did not require literacy, that is, the ability to read or write classical Chinese, the language of the elite, which was often acquired by memorizing the Confucian classics. Indeed, the two skills were likely for the most part mutually exclusive: it seems unlikely that many with the opportunity and status that literacy afforded would have spent the considerable time required to become expert at counting rod calculations; and for the illiterate, the use of counting rods can be thought of as a simple "language game" that provided a means for writing numbers and performing calculations in a particularly simple and intuitive manner.[12]

The only early records of *fangcheng* practices that have survived are extant mathematical treatises written in linear, one-dimensional narrative form, in classical Chinese: there are no two-dimensional diagrams in these early treatises; no other known sources, for example charts or diagrams, have survived; and in addition, as noted above, there are no modern practitioners. Early extant treatises provide few details: only brief, cryptic, and sometimes corrupt statements have been preserved. For example, this section will focus on problem 18 from "*Fangcheng*," chapter 8 of the *Nine Chapters*. The original text of the *Nine Chapters* preserves a written record of only the statement of the problem, the values found for the solution, and the instruction that

[10] For background on the history of Chinese mathematics, the two standard references are Jean-Claude Martzloff, *A History of Chinese Mathematics* (New York: Springer, 2006) and Li Yan and Du Shiran, *Chinese Mathematics: A Concise History*, trans. John N. Crossley and Anthony W.-C. Lun (Oxford: Clarendon Press, 1987). For background on *fangcheng*, see Hart, *Chinese Roots of Linear Algebra*; Shen, Lun, and Crossley, *Nine Chapters*. On counting board operations, see also Lay-Yong Lam and Tian-Se Ang, *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, rev. ed. (River Edge, NJ: World Scientific, 2004).

[11] Common examples of this kind of motor learning or procedural memory include typing, playing a piano, and various sports, to name only a few.

[12] On "language games," see Ludwig Wittgenstein, *Philosophical Investigations*, trans. G. E. M. Anscombe (Oxford: Blackwell, 1958). For information on counting rods, see Alexei Volkov, "Mathematics and Mathematics Education in Traditional Vietnam," in *The Oxford Handbook of the History of Mathematics*, ed. Eleanor Robson and Jacqueline A. Stedall (Oxford: Oxford University Press, 2009), 153–76.

the problem is to be solved by the *fangcheng* procedure together with the procedure for positive and negative numbers:

> Now it is given: 9 *dou*[13] of hemp, 7 *dou* of wheat, 3 *dou* of legumes, 2 *dou* of beans, and 5 *dou* of millet, [together] are valued at 140 coins; 7 *dou* of hemp, 6 *dou* of wheat, 4 *dou* of legumes, 5 *dou* of beans, and 3 *dou* of millet, [together] are valued at 128 coins; 3 *dou* of hemp, 5 *dou* of wheat, 7 *dou* of legumes, 6 *dou* of beans, and 4 *dou* of millet, [together] are valued at 116 coins; 2 *dou* of hemp, 5 *dou* of wheat, 3 *dou* of legumes, 9 *dou* of beans, and 4 *dou* of millet, [together] are valued at 112 coins; 1 *dou* of hemp, 3 *dou* of wheat, 2 *dou* of legumes, 8 *dou* of beans, and 5 *dou* of millet, [together] are valued at 95 coins.
>
> It is asked: What is the value of one *dou* [for each of the grains]?
>
> Answer: 7 coins for one *dou* of hemp; 4 coins for one *dou* of wheat; 3 coins for one *dou* of legumes; 5 coins for one *dou* of beans; 6 coins for one *dou* of millet.
>
> Procedure: Follow *fangcheng*; use the procedure for positives and negatives.
>
> 今有麻九斗、麥七斗、菽三斗、荅二斗、黍五斗，直錢一百四十；麻七斗、麥六斗、菽四斗、荅五斗、黍三斗，直錢一百二十八；麻三斗、麥五斗、菽七斗、荅六斗、黍四斗，直錢一百一十六；麻二斗、麥五斗、菽三斗、荅九斗、黍四斗，直錢一百一十二；麻一斗、麥三斗、菽二斗、荅八斗、黍五斗，直錢九十五。問一斗直幾何。答曰：麻一斗七錢，麥一斗四錢，菽一斗三錢，荅一斗五錢，黍一斗六錢。術曰：如方程，以正負術入之。[14]

Although only fragmentary written records remain, these records are sufficient to reconstruct at least the main features of the *fangcheng* procedure as a mathematical practice. That is, although the original text of the *Nine Chapters* preserves only an outline of the *fangcheng* procedure, as applied to a simpler example, problem 1, there is enough information preserved in the *Nine Chapters* to reconstruct the procedure itself in its general form.[15] This section will use this reconstruction of the *fangcheng* procedure to show how, in addition to basic counting rod operations, all that was required to solve complex linear algebra problems such as problem 18 was knowledge of a few simple patterns on the two-dimensional counting board.

Fangcheng problems were arranged using counting rods in a rectangular array, in a manner identical to the augmented matrix familiar from modern linear algebra, if we allow for differences in writing. The following diagram shows how problem 18 is laid out using counting rods on the counting board:

[13] A unit of volume, equal to ten *sheng* 升. Currently, one *dou* is equal to one decaliter.

[14] *Jiuzhang suanshu*, *juan* 8, 18b–19a.

[15] For mathematical reconstructions of the *fangcheng* procedure, see Hart, *Chinese Roots of Linear Algebra*; Shen, Lun, and Crossley, *Nine Chapters*; Martzloff, *History of Chinese Mathematics*.

𝍠	𝍡	𝍢	𝍦	𝍨
𝍢	𝍤	𝍤	𝍥	𝍦
𝍡	𝍢	𝍦	𝍣	𝍢
𝍧	𝍨	𝍥	𝍤	𝍡
𝍤	𝍣	𝍣	𝍢	𝍤
𝍱 𝍤	𝍠 𝍩 𝍡	𝍠 𝍩 𝍥	𝍠 𝍪 𝍧	𝍠 𝍬

The only difference between this and the modern mathematical notation is the orientation of the array, which corresponds to differences in writing (writing in imperial China proceeded from top to bottom, then right to left; modern English is written from left to right, then top to bottom). Written in modern mathematical notation as an augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 7 & 6 & 4 & 5 & 3 & 128 \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix} \tag{1}$$

This problem can also be written out as a system of n linear equations in n unknowns, which is probably the form most familiar to modern readers:

$$9x_1 + 7x_2 + 3x_3 + 2x_4 + 5x_5 = 140 \tag{2}$$
$$7x_1 + 6x_2 + 4x_3 + 5x_4 + 3x_5 = 128 \tag{3}$$
$$3x_1 + 5x_2 + 7x_3 + 6x_4 + 4x_5 = 116 \tag{4}$$
$$2x_1 + 5x_2 + 3x_3 + 9x_4 + 4x_5 = 112 \tag{5}$$
$$x_1 + 3x_2 + 2x_3 + 8x_4 + 5x_5 = 95. \tag{6}$$

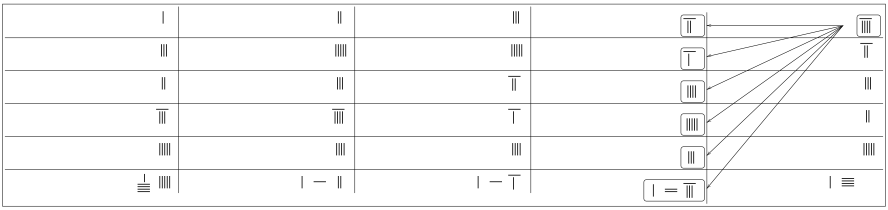
It should noted, however, that in modern linear algebra, the simpler augmented matrix (1), which corresponds to the counting board representation, is much preferred over the more cumbersome use of x_n to denote the unknowns (that is, in this case, x_1, x_2, x_3, x_4, x_5) in equations (2–6). In the following analysis, I will use the preferred modern notation, the augmented matrix.

The Fangcheng *Procedure as Simple Visual Patterns*

Solving *fangcheng* problems requires as a prerequisite a knowledge of counting board operations for addition, subtraction, multiplication, and division of integers (including negative numbers). It also requires a knowledge of the corresponding operations for fractions, although fractions rarely appear. Given knowledge of these operations, the *fangcheng* procedure is perhaps best understood as patterns on the counting board that are applied repeatedly in order to solve the problem. Extant treatises provide specific names for only two of these patterns: (1) Cross-multiplication (*biancheng* biancheng 徧乘), the multiplication of an entire column of entries by one entry of another column; (2) Term-by-term subtraction (*zhi chu* 直除, which literally means "directly subtract"),[16] the term-by-term subtraction of the entries of one column from those of another. These patterns are difficult to explain clearly in words; they are perhaps most easily understood by observing operations on the counting board. In addition, there are other simple patterns, which are not provided names, that are even more difficult to explain in words. These patterns are illustrated below.

STEP 0: Laying out the array. Before solving problem 18, we must first represent it as an array of numbers on the counting board (as shown previously on page 293).

STEP 1. Cross-multiplication (*biancheng* 徧乘). The first step is to cross-multiply, that is, to multiply each entry in the second column by the first entry of the first column. More specifically, the column $(7,6,4,5,3,128)$ is multiplied by 9, as is indicated in the following diagram:[17]



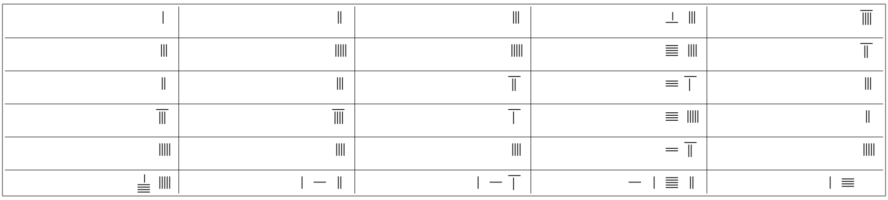
Written in modern mathematical notation, this is simply the augmented matrix modified by what is called a row operation, that is, by multiplying the entire second row by the pivot in the first row, as shown in the augmented matrix below:

$$\begin{bmatrix} \boxed{9} & 7 & 3 & 2 & 5 & 140 \\ \boxed{7} & \boxed{6} & \boxed{4} & \boxed{5} & \boxed{3} & \boxed{128} \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

[16] Alternatively, Martzloff translates this as "direct reduction." Martzloff, *History of Chinese Mathematics*, 253.

[17] The counting board, as shown in this diagram, and all of the following diagrams, is used to register the results of all of the calculations. In practice, the addition, multiplication, subtraction, or division of two individual entries may have been calculated in the head or by procedural memory; it is also possible that recording intermediary results may have been necessary. Written records of *fangcheng* practices do not describe these details. An additional space, perhaps off to one side of the counting board area, may have been used as temporary registers. For example, if two numbers being multiplied together are large (here they are not), then using counting rods to calculate the result might require a third register to store the intermediary results. In this diagram, and the diagrams that follow, I have diagrammed only the results, such as here, of the multiplication of entries. I have not shown the intermediary counting board operations involved in each individual multiplication of two entries.

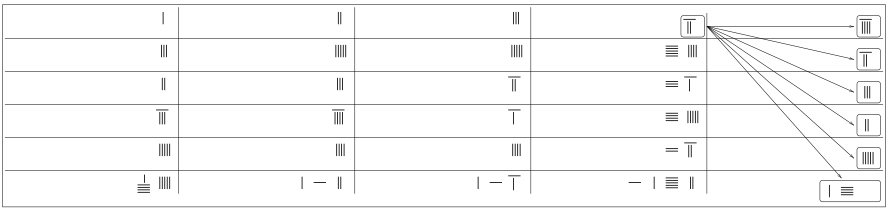
The result of the multiplication is $9(7,6,4,5,3,128) = (63,54,36,45,27,1152)$, which, on the counting board, is arranged as follows:



In the augmented matrix, we write the result as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 63 & 54 & 36 & 45 & 27 & 1152 \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

STEP 2. Cross-multiplication. The second step is another cross-multiplication, which is similar to the previous step. We multiply the entire first column by the first entry in the second column, that is, we multiply $(9,7,3,2,5,140)$ by 7, as follows:[18]

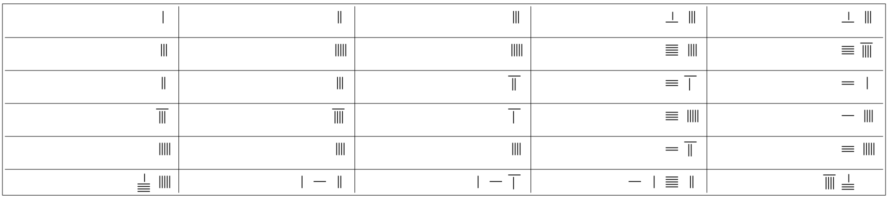


In the augmented matrix, this multiplication is represented as follows:

$$\begin{bmatrix} \boxed{9} & \boxed{7} & \boxed{3} & \boxed{2} & \boxed{5} & \boxed{140} \\ \boxed{7} & 54 & 36 & 45 & 27 & 1152 \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

[18] Again, written records of *fangcheng* practices do not fully explain the arrangement on the counting board of all the counting rods that might have been necessary at this point. In this cross-multiplication, we use the original entry in the first row of the second column, namely 7, to multiply each of the entries in the first column. I have therefore restored the original entry, 7, to show this multiplication. In the next step, the term-by-term subtraction of one column from another, we must use the modified value for the first entry of the second column, namely 63. So it is possible that there is a register outside the counting board for preserving both values. To my knowledge, no extant written records explicitly describe these temporary registers.

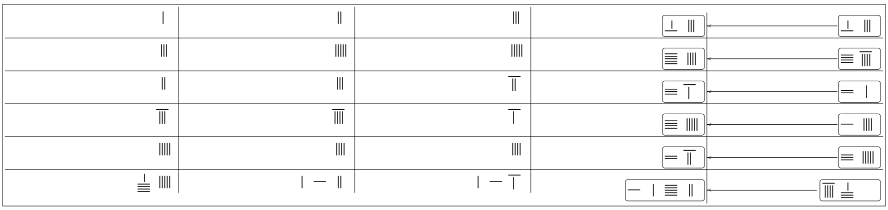
The result is then $7(9,7,3,2,5,140) = (63,49,21,14,35,980)$, which is displayed on the counting board as follows:



In the augmented matrix, the result is the following:

$$\begin{bmatrix} 63 & 49 & 21 & 14 & 35 & 980 \\ 63 & 54 & 36 & 45 & 27 & 1152 \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

STEP 3. Term-by-term subtraction (*zhi chu* 直除). In the third step, we subtract, term-by-term, the entries in the first column from those in the second column, that is, $(63,54,36,45,27,1152) - (63,49,21,14,35,980)$, which, on the counting board, again follows a very simple pattern, as can be seen from the diagram below:

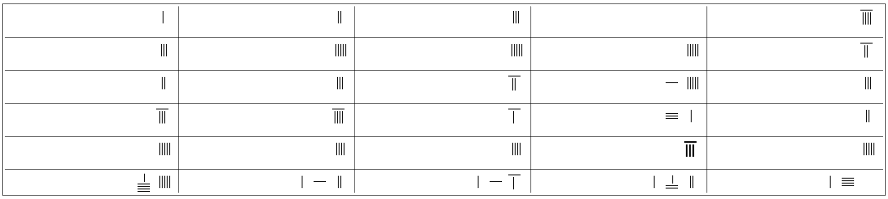


On the corresponding augmented matrix, this pattern is as follows:

$$\begin{bmatrix} \boxed{63} & \boxed{49} & \boxed{21} & \boxed{14} & \boxed{35} & \boxed{980} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \boxed{63} & \boxed{54} & \boxed{36} & \boxed{45} & \boxed{27} & \boxed{1152} \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

The result $(63,54,36,45,27,1152) - (63,49,21,14,35,980) = (0,5,15,31,-8,172)$ is represented on the counting board as shown in the following diagram (here and throughout this article I will use bold black to denote negative counting rods):[19]

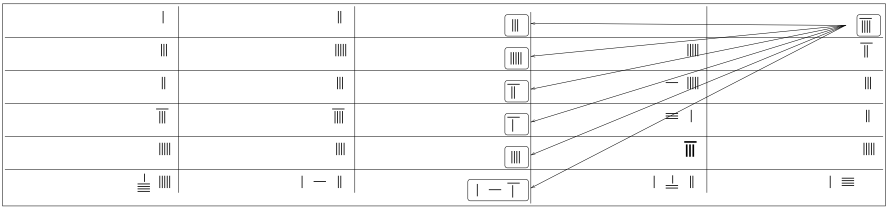
[19] At this point, as shown in the diagram, the original values of the first column, $(9,7,3,2,5,140)$, are restored. Again, extant treatises do not fully explain these details.



Again, in modern mathematical notation, this result is precisely the augmented matrix after the first step of Gaussian elimination, where we have eliminated the first entry in the second row:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

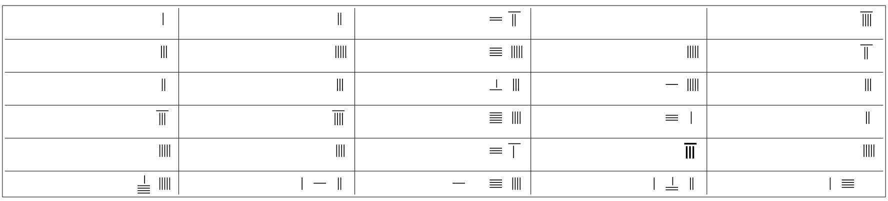
STEP 4. Cross-multiplication. The next step is to eliminate the first entry of the third column, which is done using patterns similar to those used to eliminate the first entry of the second column. We begin by multiplying the entire third column, $(3,5,7,6,4,116)$, by the first entry in the first column, 9:



On the corresponding augmented matrix, the multiplication is as follows:

$$\begin{bmatrix} \boxed{9} & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ \boxed{3} & \boxed{5} & \boxed{7} & \boxed{6} & \boxed{4} & \boxed{116} \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

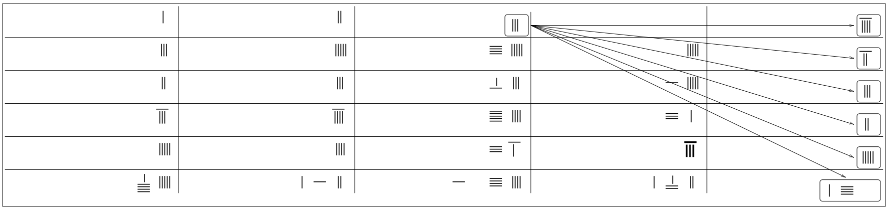
This gives the result $9(3,5,7,6,4,116) = (27,45,63,54,36,1044)$:



In the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 27 & 45 & 63 & 54 & 36 & 1044 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

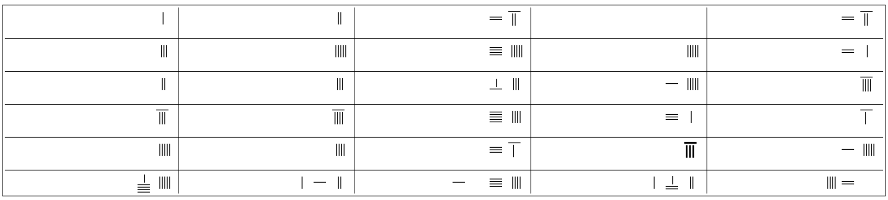
STEP 5. Cross-multiplication. We now multiply the entire first column by the first entry of the third column, namely, $3(9,7,3,2,5,140)$, as shown in the following diagram:



On the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} \boxed{9} & \boxed{7} & \boxed{3} & \boxed{2} & \boxed{5} & \boxed{140} \\ 0 & 5 & 15 & 31 & -8 & 172 \\ \boxed{3} & 45 & 63 & 54 & 36 & 1044 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

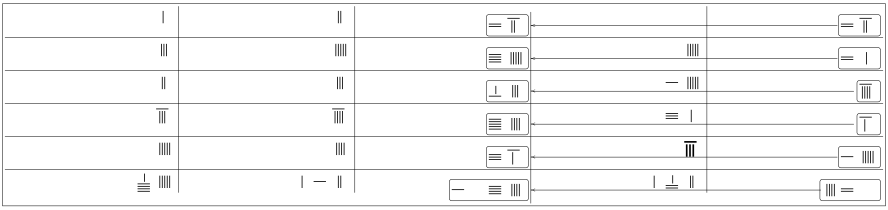
The result, $3(9,7,3,2,5,140) = (27,21,9,6,15,420)$, is displayed as follows:



On the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} 27 & 21 & 9 & 6 & 15 & 420 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 27 & 45 & 63 & 54 & 36 & 1044 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

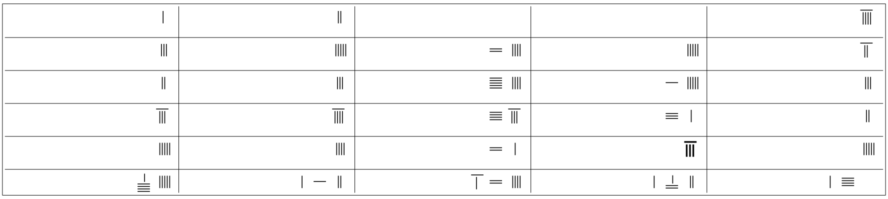
STEP 6. Term-by-term subtraction. The next step is to subtract the first column from the third column, namely, $(27,45,63,54,36,1044) - (27,21,9,6,15,420)$, as shown in the diagram below:



On the corresponding augmented matrix, the term-by-term subtraction is as follows:

$$\begin{bmatrix} \boxed{27} & \boxed{21} & \boxed{9} & \boxed{6} & \boxed{15} & \boxed{420} \\ 0 & 5 & 15 & 31 & -8 & 172 \\ \boxed{27} & \boxed{45} & \boxed{63} & \boxed{54} & \boxed{36} & \boxed{1044} \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

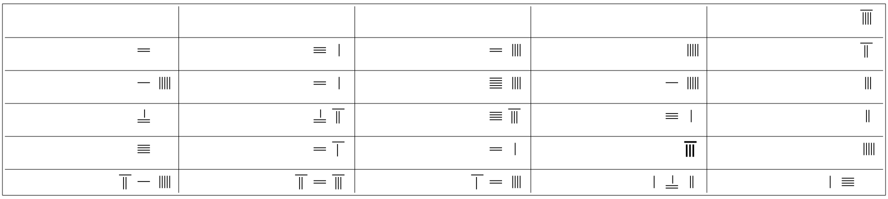
The result, $(27,45,63,54,36,1044) - (27,21,9,6,15,420) = (0,24,54,48,21,624)$, is arranged as follows:



On the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 24 & 54 & 48 & 21 & 624 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

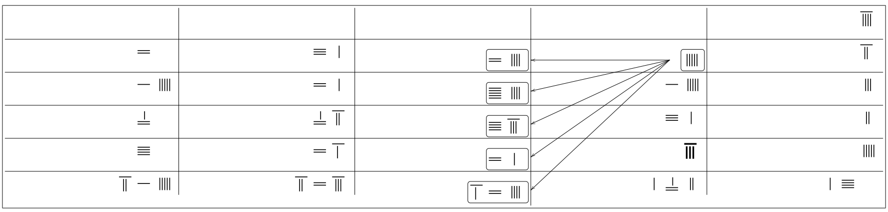
STEPS 7–12. Cross-multiplication and term-by-term subtraction. We proceed in the same manner, repeatedly using these two simple patterns, cross-multiplication and term-by-term subtraction, until the first entries of each of the remaining columns are eliminated. In this case, on the counting board, the result is displayed as follows:



On the augmented matrix, this corresponds to the elimination of the first entry in the remaining rows, as is familiar from Gaussian elimination:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 24 & 54 & 48 & 21 & 624 \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

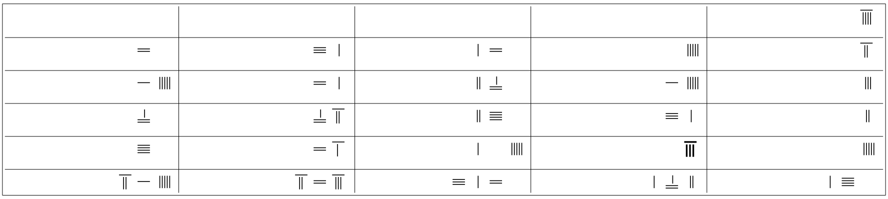
STEP 13. Cross-multiplication. Next, using these same simple patterns, we proceed to eliminate the second entries. We cross-multiply the remaining entries of the third column by the second entry of the second column, namely, $5(24, 54, 48, 21, 624)$, as shown in the following diagram:



On the corresponding augmented matrix, this is the multiplication of the third row by the pivot in the second row, as shown below:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & \boxed{5} & 15 & 31 & -8 & 172 \\ 0 & \boxed{24} & \boxed{54} & \boxed{48} & \boxed{21} & \boxed{624} \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

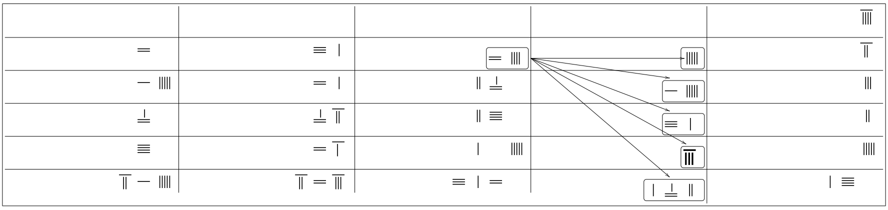
The result, $5(24, 54, 48, 21, 624) = (120, 270, 240, 105, 3120)$, is displayed as follows:



On the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 120 & 270 & 240 & 105 & 3120 \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

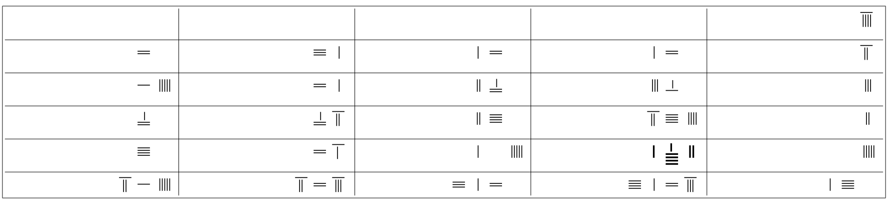
STEP 14. Cross-multiplication. Next we multiply the remaining entries of the second column by the first entry in the third column, $24(5, 15, 31, -8, 172)$, as shown in the following diagram:



On the corresponding augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & \boxed{5} & \boxed{15} & \boxed{31} & \boxed{-8} & \boxed{172} \\ 0 & \boxed{24} & 270 & 240 & 105 & 3120 \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

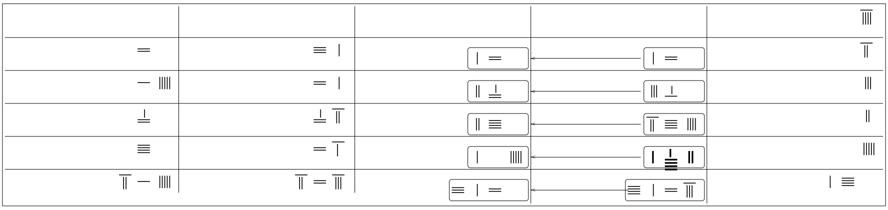
The result is $24(5, 15, 31, -8, 172) = (120, 360, 744, -192, 4128)$:



The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 120 & 360 & 744 & -192 & 4128 \\ 0 & 120 & 270 & 240 & 105 & 3120 \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

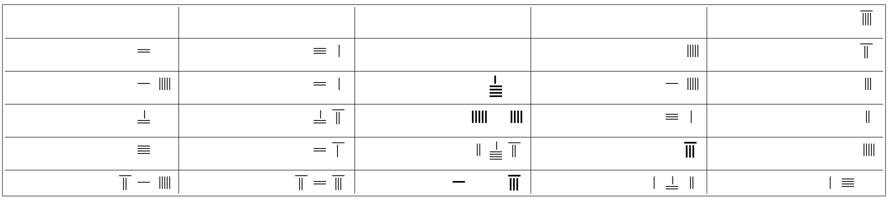
STEP 15. Term-by-term subtraction. We subtract the modified second column $(120, 270, 240, 105, 3120)$ from the modified third column $(120, 270, 240, 105, 3120)$ as follows:



On our augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & \boxed{120} & \boxed{360} & \boxed{744} & \boxed{-192} & \boxed{4128} \\ 0 & \boxed{120} & \boxed{270} & \boxed{240} & \boxed{105} & \boxed{3120} \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

This gives the result, $(120, 270, 240, 105, 3120) - (120, 360, 744, -192, 4128) = (0, -90, -504, 297, -1008)$:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 31 & 21 & 77 & 26 & 728 \\ 0 & 20 & 15 & 70 & 40 & 715 \end{bmatrix}$$

STEPS 16–21. Continued cross-multiplication and term-by-term subtraction for the second column. As was the case for the first column, we continue using the same pattern of cross-multiplication and term-by-term subtraction to remove entries in the second column. After completing the eliminations, we arrive at the following:

				𝍨
			𝍤	𝍦
𝍡𝍪𝍤	𝍢𝍮	𝍱	𝍩𝍤	𝍢
𝍡𝍯	𝍤𝍯𝍥	𝍤 𝍣	𝍫𝍠	𝍡
𝍢𝍮	𝍢𝍯𝍧	𝍡𝍱𝍦	𝍧	𝍤
𝍠𝍫𝍤	𝍩𝍥𝍱𝍡	𝍩 𝍧	𝍠𝍯𝍡	𝍠𝍬

In modern mathematical notation, in the augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & -360 & -576 & 378 & -1692 \\ 0 & 0 & -225 & -270 & 360 & 135 \end{bmatrix}$$

STEPS 22–30. Continued cross-multiplication and term-by-term subtraction for the remaining columns. We then eliminate the entries in the remaining columns that lie above the diagonal entries, proceeding as we did for the first and second columns. That is, by simply continuing to use the same pattern of cross-multiplication and term-by-term subtraction, we eliminate the entries in the remaining columns, arriving at the following:

				𝍨
			𝍤	𝍦
		𝍱	𝍩𝍤	𝍢
	𝍩𝍡𝍱𝍥	𝍤 𝍣	𝍫𝍠	𝍡
𝍪 𝍫𝍢𝍱𝍠	𝍦𝍪𝍨	𝍡𝍱𝍦	𝍧	𝍤
𝍠𝍪𝍡 𝍢𝍬𝍥	𝍪𝍠 𝍥	𝍩 𝍧	𝍠𝍯𝍡	𝍠𝍬

In modern mathematical notation, in the augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & 0 & -1296 & 729 & -2106 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

This is just the triangular form familiar from modern linear algebra. We have completed the first half of the solution, namely, forward substitution. It should be noted that, as is the case with Gaussian elimination, this procedure can be applied to an array of any size.

Back substitution as Simple Visual Patterns

If the first part of the *fangcheng* procedure is familiar from modern linear algebra, the second part, which in modern linear algebra is called "back substitution," proceeds in a counterintuitive manner, one that has hitherto not been well understood. More specifically, some modern historians of mathematics have incorrectly assimilated the *fangcheng* procedure to the modern approach to back substitution; however, several important studies of the mathematics described in the *fangcheng* procedure have noted that the approach to back substitution differs from the modern approach.[20] In *Chinese Roots of Linear Algebra*, I reconstruct this approach to demonstrate how these calculations would have been performed by following simple patterns using counting rods on a counting board. Although this approach is counterintuitive, on the counting board it is very efficient, because in general it avoids the emergence of fractions until the final step: because the counting board already uses its two dimensions to display the n conditions in n unknowns, fractions, which require at least two entries each, would considerably encumber calculations.[21] Here I provide a detailed explication of the procedure of back substitution for problem 18 using the *fangcheng* procedure on the counting board.[22] The purpose of this detailed explication of the procedure of back substitution using the *fangcheng* procedure is two-fold: first, to demonstrate how simple patterns are repeatedly applied to find the solution; second, to demonstrate the complexity of these calculations—these calculations are intimidating enough that one might hesitate to compute them without the aid of some mechanical calculating device, such as counting rods (or in my case, a computer).

STEP 31. Cross-multiplication. First we cross-multiply.[23] This pattern of cross-multiplication will be applied repeatedly to column after column, from left to right: it is first applied to the penultimate column, then to the next column to the right, then to the next, and so on, finishing with the first column on the right. The pattern of this cross multiplication, which is quite difficult to describe in words, will gradually become clear as we follow the steps for each subsequent column. In this case, the penultimate entry in the penultimate column, 729, is multiplied by the final entry in the final column, 1220346, and the final entry in the penultimate column, −2106, is multiplied by the penultimate entry in the final column, 203391, as follows:

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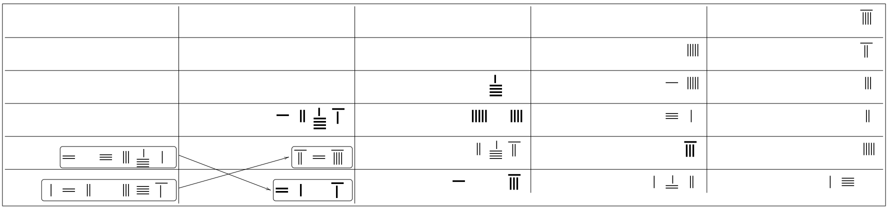
[20] In particular, see Shen, Lun, and Crossley, *Nine Chapters*.

[21] For more detail, see Hart, *Chinese Roots of Linear Algebra*.

[22] In *Chinese Roots of Linear Algebra*, I provide an example of this procedure for problem 1, which is a problem with three conditions in three unknowns. This problem, problem 18, with five conditions in five unknowns, is more complex, but follows the same patterns. To my knowledge, the approach to back substitution using the *fangcheng* procedure has hitherto never been presented in this detail.

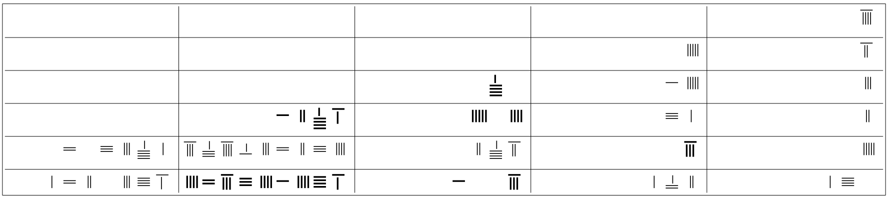
[23] This form of cross-multiplication is quite different from *biancheng*, which we saw above. To my knowledge, extant treatises do not give a name for this form of cross-multiplication.



Using modern notation, this cross-multiplication, which has no analog in modern linear algebra, is the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & 0 & -1296 & \boxed{729} & \boxed{-2106} \\ 0 & 0 & 0 & 0 & \boxed{203391} & \boxed{1220346} \end{bmatrix}$$

This gives us our result, $203391 \cdot (-2106) = -428341446$ and $1220346 \cdot 729 = 889632234$, which, on the counting board, is arranged as follows:



Written using modern notation, we have the following (it should be noted that this is no longer an augmented matrix as understood in modern mathematics):[24]

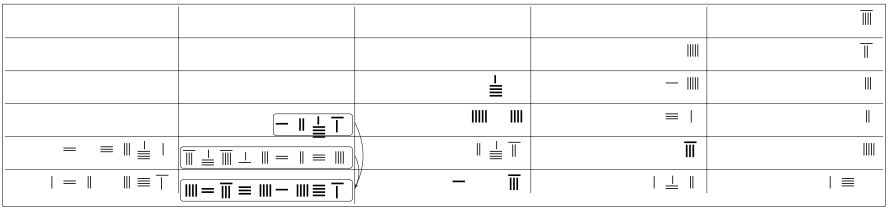
$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & 0 & -1296 & 889632234 & -428341446 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

STEP 32. Simplification.[25] We then simplify the penultimate column. In general terms, this simplification proceeds as follows: from the final entry in the column we

[24] That is, the matrix no longer corresponds to the system of linear equations given in equations (2)–(6); in modern computer science, however, arrays of numbers are often used in this manner, and need not correspond to the coefficients of a system of linear equations.

[25] Again, to my knowledge, extant treatises do not provide a name for this form of simplification.

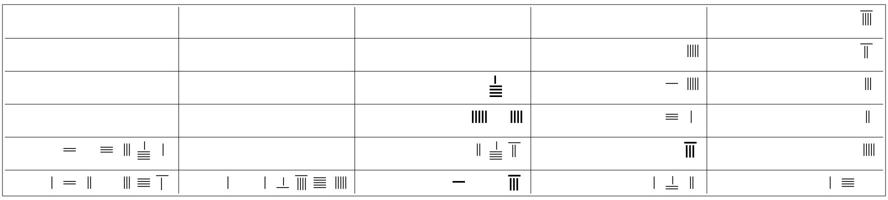
successively subtract each of the entries above it, until we reach the diagonal entry, that is, the first remaining nonzero entry in that column;[26] we then divide the result of the successive subtractions by this diagonal entry. In this case, we subtract the penultimate entry from the final entry, and then divide by the fourth entry, as shown in the following diagram:



Written using modern notation, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & 0 & \boxed{-1296} & \boxed{889632234} & \boxed{-428341446} \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

That is, we calculate $-428341446 - 889632234 = -1317973680$, and then we calculate $(-1317973680) \div (-1296) = 1016955$, with the result displayed as follows:[27]



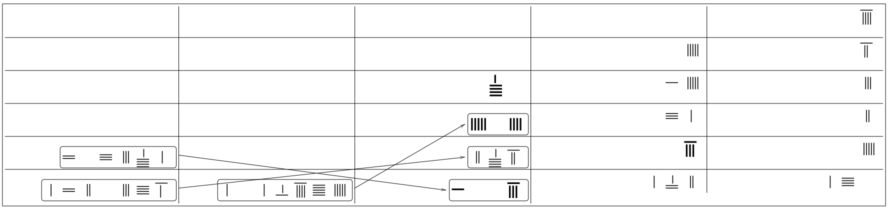
In modern notation, the result is as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -504 & 297 & -1008 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

[26] In modern linear algebra, we would call this the pivot.

[27] I have removed the remaining entries from this column. Again, there are no descriptions specifying the arrangement of the counting board at this stage.

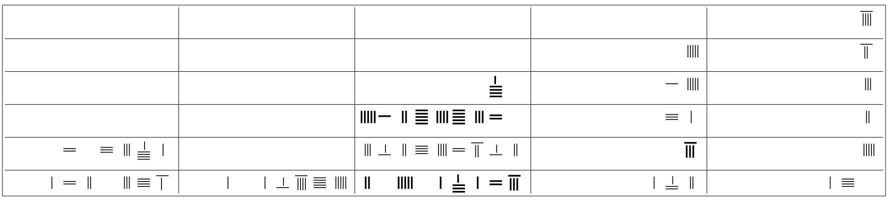
STEP 33. Cross-multiplication. Next we again cross multiply, extending the pattern used in the previous column. Here, we first multiply the final entry in the third column by the penultimate entry in the final column, $203391 \cdot (-1008)$. Then we multiply the penultimate entry in the third column by the final entry in the final column, $1220346 \cdot 297$. Then we multiply the fourth entry in the third column by the final entry in the fourth column, $1016955 \cdot (-504)$. These cross multiplications, displayed on the counting board, can be represented as follows, illustrating the simple visual pattern used:



Using modern notation, this can be represented as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & \boxed{-504} & \boxed{297} & \boxed{-1008} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1016955} \\ 0 & 0 & 0 & 0 & \boxed{203391} & \boxed{1220346} \end{bmatrix}$$

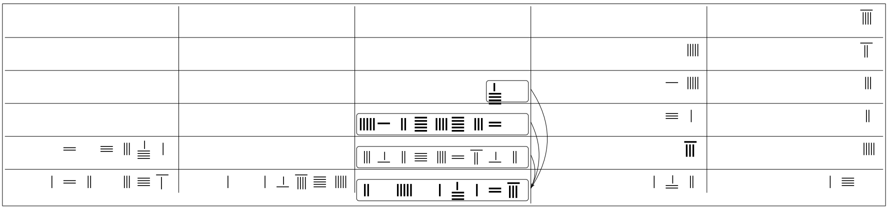
The result of these multiplications is then $203391 \cdot (-1008) = -205018128$ for the final entry in the third column, $1220346 \cdot 297 = 362442762$ for the penultimate entry, and $1016955 \cdot (-504) = -512545320$ for the fourth entry:



In modern notation, this is the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & -90 & -512545320 & 362442762 & -205018128 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

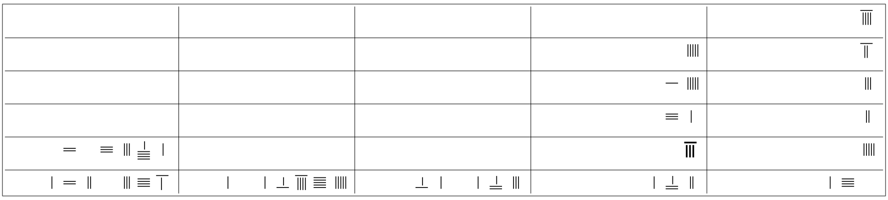
STEP 34. Simplification. The next step is simplification, extending the same pattern used in the previous simplification. We successively subtract from the final entry in this column the entries in the column above it, until we have reached the diagonal entry, the first nonzero entry in the column, and then we divide by the diagonal entry, as shown below:



In modern notation, this is the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & \boxed{-90} & \boxed{-512545320} & \boxed{362442762} & \boxed{-205018128} \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

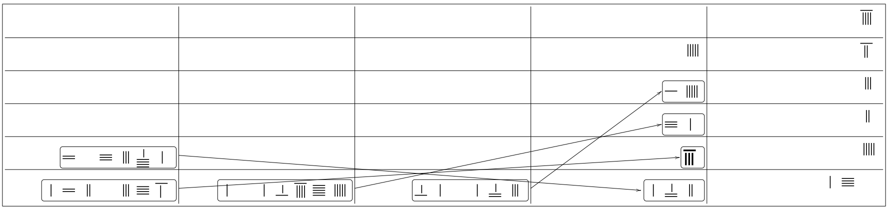
That is, we successively calculate $-205018128 - 362442762 = -567460890$, then $-567460890 - (-512545320) = -54915570$, and $(-54915570) \div (-90) = 610173$, giving the following result:



In modern notation, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 15 & 31 & -8 & 172 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

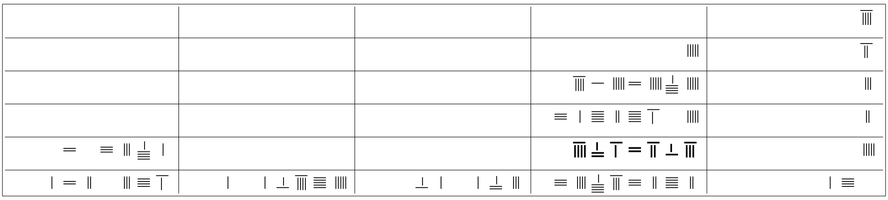
STEP 35. Cross-multiplication. We again use the previous pattern for cross multiplication: we multiply the final entry in the second column by the penultimate entry in the final column, $203391 \cdot 172$; we multiply the fifth entry in the second column by the final entry in the fifth column, $1220346 \cdot (-8)$; we multiply the fourth entry in the second column by the final entry in the fourth column, $1016955 \cdot 31$; and we multiply the third entry in the second column by the final entry in the third column, $610173 \cdot 15$. These operations again follow the same simple pattern, as can easily be seen from the diagram below:



In modern notation, these cross-multiplications appear as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & \boxed{15} & \boxed{31} & \boxed{-8} & \boxed{172} \\ 0 & 0 & 0 & 0 & 0 & \boxed{610173} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1016955} \\ 0 & 0 & 0 & 0 & \boxed{203391} & \boxed{1220346} \end{bmatrix}$$

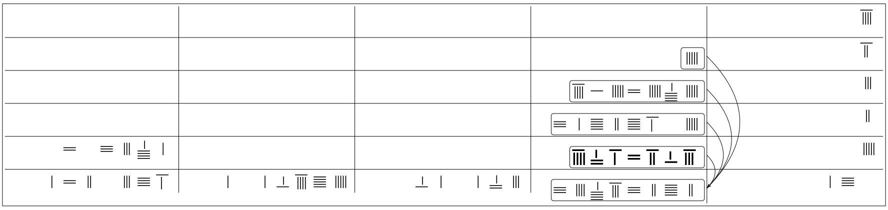
The result, recorded on the counting board, is as follows:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 5 & 9152595 & 31525605 & -9762768 & 34983252 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

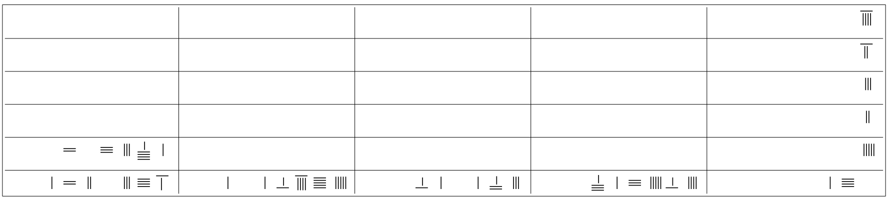
STEP 36. Simplification. Again, we extend the same pattern to simplify the second column, repeatedly subtracting from the final entry the entries above it until we reach the entry on the diagonal, namely, the second entry, 5. We then use the second entry, 5, to divide the result of the successive subtractions, which is stored in the position of the final entry. On the counting board, we have the following:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & \boxed{5} & \boxed{9152595} & \boxed{31525605} & \boxed{-9762768} & \boxed{34983252} \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

That is, we successively calculate $34983252-(-9762768)=44746020$, then next $44746020-31525605=13220415$, then next $13220415-9152595=4067820$, and finally $4067820\div 5=813564$, which is displayed as follows:

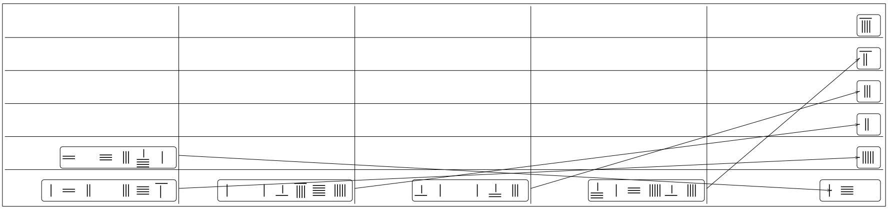


In modern notation, we have the following:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 0 & 0 & 0 & 0 & 813564 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

STEP 37. Cross-multiplication. The final cross multiplication again simply further extends the same pattern as in the previous steps. We first multiply the final entry in the first column by the penultimate entry in the final column, $203391 \cdot 140$; we multiply the fifth entry in the first column by the final entry in the fifth column,

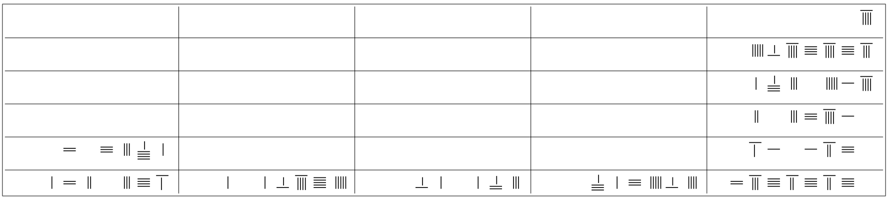
$1220346 \cdot 5$; we multiply the fourth entry in the first column by the final entry in the fourth column, $1016955 \cdot 2$; we multiply the third entry in the first column by the final entry in the third column, $610173 \cdot 3$; and we multiply the second entry in the first column by the final entry in the second column, $813564 \cdot 7$. The pattern of this cross multiplication on the counting board is as follows:



In modern notation, these cross-multiplications follow the same simple pattern, as shown below:

$$\begin{bmatrix} 9 & \boxed{7} & \boxed{3} & \boxed{2} & \boxed{5} & \boxed{140} \\ 0 & 0 & 0 & 0 & 0 & \boxed{813564} \\ 0 & 0 & 0 & 0 & 0 & \boxed{610173} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1016955} \\ 0 & 0 & 0 & 0 & \boxed{203391} & \boxed{1220346} \end{bmatrix}$$

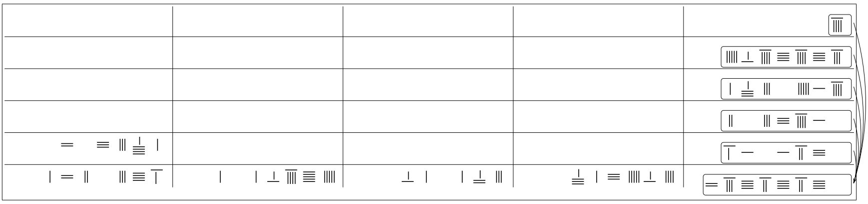
The result of the cross-multiplication, recorded on the counting board, is as follows:



In modern notation, we have the following:

$$\begin{bmatrix} 9 & 5694948 & 1830519 & 2033910 & 6101730 & 28474740 \\ 0 & 0 & 0 & 0 & 0 & 813564 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

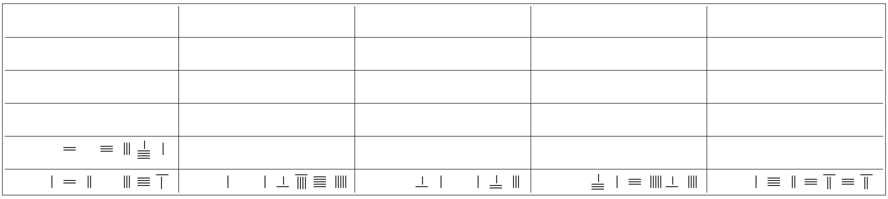
STEP 38. Simplification. The simplification of the first column again simply extends the same pattern used in the previous steps. From the final entry we successively subtract the entries above it until we have arrived at the entry on the diagonal, in this case the first entry, 9, which is used to divide the result.



In modern notation, we have the following:

$$\begin{bmatrix} \boxed{9} & \boxed{5694948} & \boxed{1830519} & \boxed{2033910} & \boxed{6101730} & \boxed{28474740} \\ 0 & 0 & 0 & 0 & 0 & 813564 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

That is, we subtract the fifth entry, $28474740 - 6101730 = 22373010$; we subtract the fourth entry, $22373010 - 2033910 = 20339100$; we subtract the third entry, $20339100 - 1830519 = 18508581$; and we subtract the second entry, $18508581 - 5694948 = 12813633$. Then we then divide the result by the diagonal entry, giving $12813633 \div 9 = 1423737$:



In modern notation, we have the following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1423737 \\ 0 & 0 & 0 & 0 & 0 & 813564 \\ 0 & 0 & 0 & 0 & 0 & 610173 \\ 0 & 0 & 0 & 0 & 0 & 1016955 \\ 0 & 0 & 0 & 0 & 203391 & 1220346 \end{bmatrix}$$

STEP 39. Final division. The final step is division, which can be understood as simplifying fractions. We divide the final entry in each column by the penultimate entry in the final column. That is, we divide the final entry in the first column, $1423737 \div 203391 = 7$; we divide the final entry in the second column, $813564 \div 203391 = 4$; we divide the final entry in the third column, $610173 \div 203391 = 3$; we divide the final entry in the fourth column, $1016955 \div 203391 = 5$; and we divide the final entry in this fifth column, $1220346 \div 203391 = 3$. The result of this final division is the solution to the problem:

𝍥	𝍤	𝍢	𝍣	𝍦

In modern notation, we have the following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

In sum, the *fangcheng* procedure, as described in the *Nine Chapters*, outlines what is now known to be the most powerful and general method known for solving systems of linear equations, namely, what is now called Gaussian elimination. We might summarize some of the main features of the *fangcheng* procedure as follows:

1. Prerequisites: The only prerequisites are facility with addition, subtraction, multiplication, division, and fractions on the counting board.
2. Patterns: Beyond the above prerequisites, all that is required is an understanding of simple patterns that are repeatedly applied. There are two basic patterns that are followed during the process of elimination: (i) cross-multiplication (*biancheng* 徧乘); and (ii) term-by-term subtraction (*zhi chu* 直除). There are also two additional simple patterns that are followed in the process of back substitution (neither of which is easy to describe in words, and neither of which is given a name in extant treatises): (i) cross-multiplication; and (ii) simplification. (Again, these latter two patterns can best be understood through the diagrams provided.)
3. Generality: The *fangcheng* procedure—by using these simple patterns—is completely general, and can be used (with some slight modifications) to solve any system of *n* conditions in *n* unknowns for which a solution exists.
4. Sophistication: The approach used for back substitution indicates considerable sophistication. It is much more sophisticated than the usual intuitive approach familiar from modern linear algebra.
5. Efficiency: These calculations, which are all rote applications of counting-rod operations and simple patterns, could likely be calculated, quickly, efficiently, and with little effort.
6. Literacy: Nothing in the *fangcheng* procedure requires literacy, and in fact literacy could contribute little to mastering this practice.

Written Records of *Fangcheng* Practices

Although the *fangcheng* procedure is easily learned by observing two-dimensional patterns on the counting board, as noted above, it is difficult to describe using words, that is, through the medium of a one-dimensional narrative. In fact, as recorded in extant Chinese mathematical treatises, *fangcheng* practices are rendered virtually unintelligible—to understand the text, one must first understand the practice.

Extant mathematical treatises suggest that those who recorded *fangcheng* practices in writing, often to present to the imperial court, understood only the rudiments of these practices, and they often express contempt for its practitioners. The example I will present here is from the earliest extant record of *fangcheng* practices, the *Nine Chapters*. Liu Hui 劉徽 (fl. 263 CE), to whom the earliest extant commentary on the *Nine Chapters* is conventionally attributed, is usually considered the most eminent mathematician of imperial China. Yet, as I note in *Chinese Roots of Linear Algebra*, the commentary attributed to him evinces little understanding of the fact that the complexity of the *fangcheng* procedure in the *Nine Chapters* was necessary in order to avoid even more complicated calculations with fractions. In fact, the commentary expresses derision toward practitioners for their rote application of the arcane *fangcheng* procedure:

> Those who are clumsy in the essential principles vainly follow this original [*fangcheng*] procedure, some placing counting rods so numerous that they fill a carpet, seemingly so fond of complexity as to easily make mistakes. They seem to be unaware of the error [in their approach], and on the contrary, desire by the use of more [counting rods] to be highly esteemed. Therefore, of their calculations, all are ignorant of the establishment of understanding; instead, they are specialized to an extreme.
>
> 其拙于精理從按本術者，或用算而布氊，方好煩而喜誤，曾不知其非，反欲以多為貴。故其算也，莫不（同）〔闇〕于設通而專于一端。[28]

The commentary follows this accusation with several alternative solutions to problem 18. These solutions, as recorded in the commentary, seem to confirm that the writer understood only the rudiments of solving *fangcheng* problems. Here I will examine in detail one of these solutions.

The "Old Method"

The commentary in the *Nine Chapters* presents what is asserted to be the "old method" (*jiu shu* 舊術),[29] which is sometimes assumed to be the *fangcheng* pro-

[28] *Jiuzhang suanshu*, *juan* 8, 19a; Shen, Lun, and Crossley, *Nine Chapters*, 426; Chemla and Guo, *Neuf chapitres*, 650–51.

[29] *Jiu* 舊 is translated as "original" in Shen, Lun, and Crossley, *Nine Chapters*, 427. I believe the term is pejorative, so I have translated *jiu* as "old."

cedure.[30] The commentary, however, fails to follow the sophisticated algorithm of the *fangcheng* procedure as presented in the original text of the *Nine Chapters*. From a computational standpoint, compared with the *fangcheng* procedure, the approach taken in the commentary has several disadvantages: it does not follow an algorithm; no general method is presented; and thus it cannot be rapidly computed at each step. But most important, if the approach taken in the commentary is applied to general cases (that is, beyond the 18 contrived problems recorded in the *Nine Chapters*), fractions will often appear in back substitution, and fractions, each of which requires two dimensions on the counting board, would considerably encumber *fangcheng* calculations, which already employ the two dimensions to record the array. In this sense, the approach presented in the commentary might at first appear to be serendipitous. I will argue that what the commentary calls the "old method" is a corrupt record of a recondite solution that attempts to minimize the number of counting rods used. What is indisputable, however, is that the commentary does not address the considerable computational advantages of the rote application of the sophisticated *fangcheng* algorithm.

The approach taken in the commentary is difficult to reconstruct, because the text here is hopelessly corrupt. Modern attempts to reconstruct this passage—so that it makes sense mathematically—require numerous emendations, transpositions, insertions, and deletions.[31] None are satisfactory. That calculations this corrupt were copied and passed on, without correction, in all extant versions of the *Nine Chapters*, including the imperially sponsored edition, is consistent with the rudimentary grasp of the *fangcheng* procedure by the literati who compiled mathematical treatises. Even so, despite their differences, the reconstructed versions are all similar in the sense that the approach taken in the commentary does not follow any algorithm or method. As we will see, it seems that this approach is the corrupt record of a solution that requires only 77 counting rods, perhaps presented in support of the preceding criticism that the original *fangcheng* procedure used too many counting rods (185 counting rods are required to display the solution shown on pages 294–303).

The following is the "old method," as recorded in the commentary to the *Nine Chapters* (I have inserted the individual step numbers in brackets):

---

[30] The commentary also presents a "new method" *xin shu* 新術 and an "alternative method" *qi yi shu* 其一術. I cannot address these methods in this article, except to say that analysis of the "new method" and the "alternative method" yields conclusions consistent with those presented here.

[31] For a comparison of several modern attempts at reconstruction, see Li Jimin 李继闵, *Jiuzhang suanshu jiaozheng* 九章算术校证 [Nine chapters on the mathematical arts, critical edition] (Xi'an: Shanxi kexue jishu chubanshe 陕西科学技术出版社, 1993). Li notes that Guo Shuchun, one of the foremost historians of Chinese mathematics, inserts 27 characters, changes 9 characters, moves 11 characters, and deletes 1 character, changing 48 out of a total of 129 characters. Other reconstructions are similar. For a more recent attempt, see Mo Shaokui 莫绍揆, "Youguan *Jiuzhang suanshu* de yixie taolun" 有关《九章算术》的一些讨论 [Several points about the Nine chapters on the mathematical arts], *Ziran kexueshi yanjiu* 自然科学史研究 19 (2000): 97–113.

[1] First place the third column, subtract the fourth column, and subtract the third column. [2] Next,[32] place the second column, with the third column subtract from the second column, eliminating its first entry. [3] Next, place the right column and eliminate its first entry. [4] Next, with the fourth column subtract the first entry of the left column. [5] Next, with the left column, eliminate the first entry of the fourth column and the second column. [6] Next, with the fifth column subtract the first entry of the second column; the remainder can be halved. [7] Next, with the second column, eliminate the first entry of the fourth column; reduce the remainder, reduce the improper fraction to a mixed fraction, obtaining zero,[33] namely the price of millet. [8] With the divisor subtract the second column, obtaining the price of beans. [9] With the left column obtain the price of wheat; the third column is the price of hemp. [10] The right column is the price of legumes. [11] This method uses 77 counting rods.[34]

[1] 先置第三行，以減第四行，及減第三行。[2] 次置第二行，以第二行減第三行，去其頭位。[3] 次置右行去其頭位。[4] 次以第四行減左行頭位。[5] 次以左行去第四行及第二行頭位。[6] 次以第五行減第二行頭位，餘可半。[7] 次以第二行去第四行頭位，餘約之為法，實如法而一，得空即有黍價。[8] 以法減第二行，得荅價。[9] 左行得麥價，第三行麻價。[10] 右行得菽價。[11] 如此凡用七十七算。[35]

Reconstruction of the "Old Method"

Below I attempt to reconstruct the calculations recorded in the commentary on the counting board, interpreting the instructions in the commentary as charitably as possible, and following them as far as possible. As we shall see, it is only possible to follow the instructions to step 4. After step 4, it becomes increasingly difficult to reconcile the stated instructions with the previous steps, however they are interpreted.

STEP 1. "First place the third column, subtract the fourth column, and subtract the third column" (lines 1 and 2 on this page). The first part of the first step is unambiguous—place (*zhi* 置) the third column on the counting board. We then have the following:

[32] More literally, *ci* 次 means "second" (*di er* 第二). Here its repeated use is better translated as "next."

[33] That is, the improper fraction reduces to an integer, so the remaining numerator for the fraction is zero.

[34] The character *suan* 算 usually means to calculate. The usual interpretation is that 77 calculations are required. However, it is difficult to see how to count the number of calculations and arrive at the number 77. *Suan* 算 is also sometimes used interchangeably with *suan* 筭, which means counting rods. In this case, the evidence suggests that 77 *suan* refers to the number of counting rods, as does 124 *suan* in the following passage.

[35] *Jiuzhang suanshu*, *juan* 8, 20b–21a.

		𝍢		
		𝍤		
		𝍦		
		𝍥		
		𝍣		
		𝍠𝍩𝍥		

Represented as an augmented matrix, we have the following:

$$\begin{bmatrix} 3 & 5 & 7 & 6 & 4 & 116 \end{bmatrix}$$

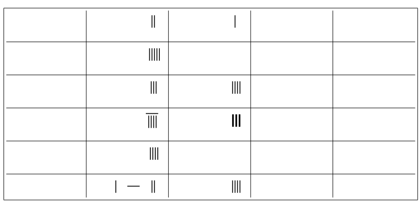
This instruction, however, stands in contrast to the *fangcheng* procedure, as outlined in the *Nine Chapters* following problem 1, in which all of the columns are placed on the counting board at the outset, from right to left. Here we are instructed to place only one column on the counting board, and not the first column but the third. The second part of the first step—the two instructions “subtract the fourth column, and subtract the third column”—also diverges from the *fangcheng* procedure in several respects. Two columns are first operated on, and only afterward are the other columns placed on the counting board. There are explicit instructions to place two of the columns in later steps: the second column is placed on the counting board in the second step (lines 2 to 4 on the preceding page), and the first (right) column is placed on the counting board in the third step (line 4 on the facing page). There are no explicit instructions to place the fourth and fifth (left) columns on the counting board—presumably they are placed on the board at the time of their first operation, namely, the first step for the fourth column (lines 1 and 2 on the preceding page), and the fifth step for the fifth column (lines 6 and 7 on the facing page). Although there is no explanation given for the delayed placement of the remaining columns, or for first operating on those columns placed on the board, it is difficult to see any other possible motive than to minimize the number of counting rods. In any case, if we place the fourth column on the counting board, the results are as follows:

	𝍡	𝍢		
	𝍤	𝍤		
	𝍢	𝍦		
	𝍨	𝍥		
	𝍣	𝍣		
	𝍠𝍩𝍡	𝍠𝍩𝍥		

On the augmented matrix, we have the following:

$$\begin{bmatrix} 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \end{bmatrix}$$

The two instructions that comprise the second part of the first step—"subtract the fourth column" and "subtract the third column"—are not clear. That is, the subtracting columns are given, but from which columns they are to be subtracted is only implied; there is no indication of whether the subtracting columns are subtracted once or whether they are subtracted multiple times (or equivalently, whether the subtracting columns are subtracted as they stand, or whether they are first multiplied term-by-term by constants); and the results of the operations are not given (for example, specifying which entries are eliminated). Without at least some of this information, we cannot be certain what these instructions mean. And again, these instructions diverge from the *fangcheng* procedure in several respects: in the *fangcheng* procedure, a multiple of the first column is subtracted from a multiple of the second column in order to eliminate the first entry in the second column, and then additional entries are similarly eliminated in the order familiar from modern linear algebra; in the *fangcheng* procedure, we do not subtract one column from another, and then subtract the latter from the former. Perhaps the most reasonably interpretation of the first instruction here is to subtract the fourth column, once, from the third column, $(3,5,7,6,4,116)-(2,5,3,9,4,112)=(1,0,4,-3,0,4)$. The commentary offers no explanation for this particular choice. Although it might at first appear that the purpose is to reduce the first entry in one of the columns to 1 and then use that column to eliminate the first entry in the remaining columns, the fifth column already has a 1 as its first entry. A more plausible explanation for this particular choice is that not one but two entries are eliminated, which is again consistent with the hypothesis that the motive is to minimize the number of counting rods. In any case, the result on the counting board is as follows:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \end{bmatrix}$$

The next instruction is even more perplexing. Apparently we are to use the modified third column to subtract term-by-term from the fourth column. Perhaps the most reasonable interpretation is that we subtract the modified third column twice from the fourth column in order to eliminate the first entry of the fourth column, even though there is no mention of subtracting twice, nor is there any mention of eliminating an entry. In any case, if we subtract the third column twice from the fourth column, $(2,5,3,9,4,112) - 2(1,0,4,-3,0,4) = (0,5,-5,15,4,104)$, the result is as follows:

		𝍠		
	𝍤			
	𝍤	𝍣		
	𝍩 𝍤	**𝍢**		
	𝍣			
	𝍠 𝍣	𝍣		

On the augmented matrix, we have the following:

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \end{bmatrix}$$

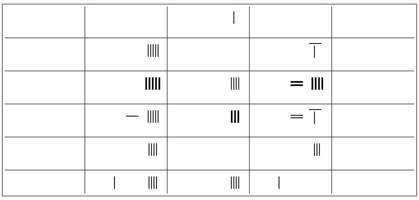
STEP 2. "Next, place the second column, with the third column subtract from the second column, eliminating its first entry" (lines 2 to 4 on page 316). This step is clear —the subtracting column, the column subtracted from, and the result are specified. This step is consistent with our results thus far. Assuming that we are to first place the second column on the counting board before we operate with it, we have the following:

		𝍠	𝍦	
	𝍤		𝍥	
	𝍤	𝍣	𝍣	
	𝍩 𝍤	**𝍢**	𝍤	
	𝍣		𝍢	
	𝍠 𝍣	𝍣	𝍠 𝍪 𝍧	

The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 7 & 6 & 4 & 5 & 3 & 128 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \end{bmatrix}$$

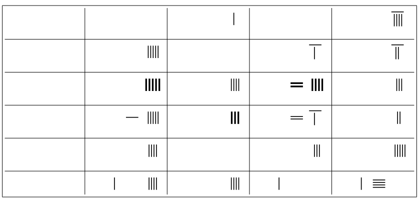
We then subtract the third column multiplied by 7 from the second column, namely, $(7,6,4,5,3,128) - 7(1,0,4,-3,0,4) = (0,6,-24,26,3,100)$, so that the first entry is eliminated. The result on the counting board is as follows:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 0 & 6 & -24 & 26 & 3 & 100 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \end{bmatrix}$$

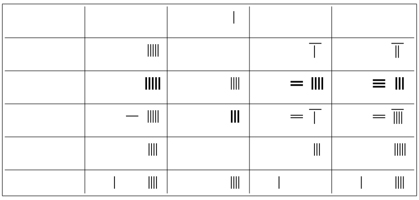
STEP 3. "Next, place the right column and eliminate its first entry" (line 4 on page 316). This instruction is reasonably clear—it specifies that we are to subtract from the first (right) column, and it states the result of the subtraction; and even though it does not specify which column we are to use to subtract, we can reasonably infer that it is the third column, multiplied by 9. Assuming this is the case, we place the first column, $(9,7,3,2,5,140)$, on the counting board giving the following result:



Our augmented matrix is as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ 0 & 6 & -24 & 26 & 3 & 100 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \end{bmatrix}$$

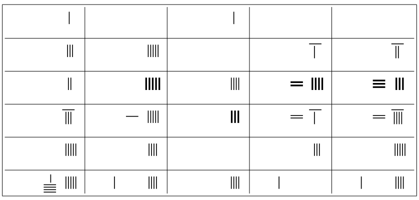
If we subtract the third column, multiplied by 9, from the first column, namely, $(9,7,3,2,5,140)-9(1,0,4,-3,0,4)=(0,7,-33,29,5,104)$, the result is as follows:



On the augmented matrix, the result is as follows:

$$\begin{bmatrix} 0 & 7 & -33 & 29 & 5 & 104 \\ 0 & 6 & -24 & 26 & 3 & 100 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \end{bmatrix}$$

STEP 4. "Next, with the fourth column subtract the first entry of the left column" (lines 5 and 6 on page 316). It is at this point that the instructions recorded in the commentary fail to make sense. If we place the fifth (left) column on the counting board, we have the following result:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 0 & 7 & -33 & 29 & 5 & 104 \\ 0 & 6 & -24 & 26 & 3 & 100 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 5 & -5 & 15 & 4 & 104 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

We can see that the instructions here cease to correspond to the calculations made so far: there is no first entry in the fourth column with which to subtract from the first entry of the fifth column. Other alternative interpretations of the previous steps fare no better.

STEP 5 … The following steps increasingly go awry, no matter how the previous steps are interpreted. As noted above, reconstructions of this problem require considerable changes to the text. As one example of a reconstruction, below is the final step from what I consider to be one of the most plausible reconstructions of the solution, that found in the translation of the *Nine Chapters* by Shen, Lun, and Crossley:[36]

		𝍠		
				𝍠
𝍤		𝍣		𝍨
𝍡		𝍢	𝍣	𝍢
𝍡	𝍫𝍠		𝍪𝍠	𝍡
𝍫𝍦	𝍠𝍯𝍥	𝍣	𝍠𝍬𝍥	𝍣

On the augmented matrix, we have the following:

$$\begin{bmatrix} 0 & 1 & -9 & 3 & 2 & 4 \\ 0 & 0 & 0 & -4 & -21 & -146 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 31 & 186 \\ 0 & 0 & 5 & 2 & 2 & 37 \end{bmatrix}$$

From this, we can see that the elimination of entries follows no particular order. No one has provided an adequate explanation for the particular choices made. No one has provided a satisfactory reconstruction of the text—in fact the various reconstructions, and hence the final configurations on the counting board, differ perhaps as much from each other as they do from the received text.

Mathematical Reconstruction of the "Old Method"

If the text is hopelessly corrupt—far beyond any possibility of textual reconstruction—we might ask if it is perhaps possible to reconstruct the solution mathematically. That is, is there a solution that proceeds in basically the same manner and provides the answer recorded in the text, 77 counting rods? In fact, there is, as will be shown below.

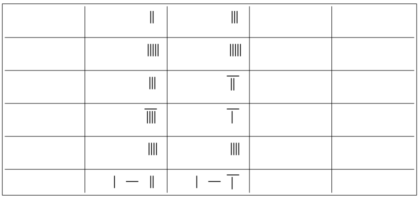
More specifically, the "old method" employs several techniques that are contrary to the *fangcheng* procedure, and for which the most plausible explanation would

[36] Shen, Lun, and Crossley offer translations and detailed mathematical explanations of the "old method" along with other methods given for solving problem 18. Here, as throughout my work, I am greatly indebted to their pioneering work on the subject. My reconstruction of this problem, however, differs from theirs. See Shen, Lun, and Crossley, *Nine Chapters*, 426–38.

seem to be that they represent an attempt to minimize the number of counting rods: (1) delaying the placement of columns; (2) operating on columns before all of the columns have been placed; (3) eliminating entries opportunistically (as opposed to the predefined order prescribed by the *fangcheng* procedure); and (4) reducing columns by dividing all of the entries by a common divisor. If we allow (3) above—that entries can be eliminated in any order—there are approximately 14,400 variations to solving problem 18. There are more variations if we count the various possibilities in (1), (2), and (4) separately. If in solving problem 18 we allow (1)–(4) above, it turns out that 77 counting rods seems to be the minimum number necessary to display at each step the results of the calculations. That this was apparently known at the time indicates considerable expertise in the thousands of possible variations to solving this *single* problem. It also suggests that whoever recorded this solution was unable to reproduce this difficult result.

The following mathematical reconstruction demonstrates how Problem 18 can be solved using only 77 counting rods to display the results at each step.

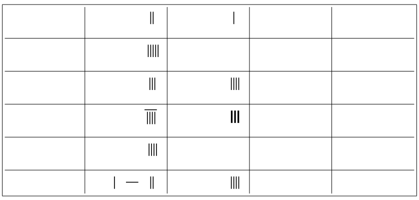
STEP 1. First, place the third and fourth columns on the counting board, using 44 counting rods:



Represented as an augmented matrix, we have the following:

$$\left[\begin{array}{cccccc} &&&&& \\ &&&&& \\ 3 & 5 & 7 & 6 & 4 & 116 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ &&&&& \end{array}\right]$$

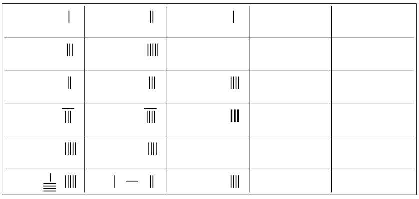
STEP 2. Next subtract the fourth column from the third column, $(3,5,7,6,4,116) - (2,5,3,9,4,112) = (1,0,4,-3,0,4)$, and display the results, using 35 counting rods:



The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \end{bmatrix}$$

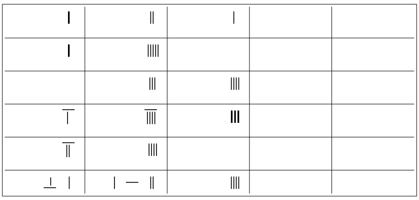
STEP 3. Next, place the fifth column on the counting board, using 60 counting rods:



Our augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ 1 & 3 & 2 & 8 & 5 & 95 \end{bmatrix}$$

STEP 4. Then subtract two times the fourth column from three times the fifth column, $3(1,3,2,8,5,95)-2(2,5,3,9,4,112)=(-1,-1,0,6,7,61)$, and display the results on the counting board, using 45 counting rods:



On the augmented matrix, we have the following:

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 5. Next place the first column on the counting board, using 68 counting rods:

𝍠	𝍡	𝍠		𝍨
𝍠	𝍤			𝍦
	𝍢	𝍣		𝍢
𝍥	𝍨	**𝍢**		𝍡
𝍦	𝍣			𝍤
𝍮𝍠	𝍠𝍩𝍡	𝍣		𝍠𝍬

The augmented matrix is as follows:

$$\begin{bmatrix} 9 & 7 & 3 & 2 & 5 & 140 \\ & & & & & \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 6. Then subtract the fourth column from the first column, $(9,7,3,2,5,140) - (2,5,3,9,4,112) = (7,2,0,-7,1,28)$, and display the results, using 60 counting rods:

𝍠	𝍡	𝍠		𝍦
𝍠	𝍤			𝍡
	𝍢	𝍣		
𝍥	𝍨	**𝍢**		**𝍦**
𝍦	𝍣			𝍠
𝍮𝍠	𝍠𝍩𝍡	𝍣		𝍪𝍧

On the augmented matrix, we have the following:

$$\begin{bmatrix} 7 & 2 & 0 & -7 & 1 & 28 \\ & & & & & \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

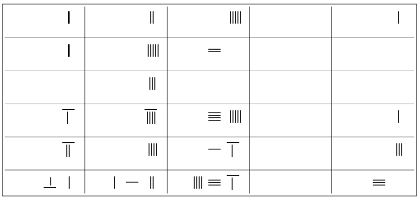
STEP 7. Next subtract two times the fifth column from the first column, and divide the result by 5, $((7,2,0,-7,1,28)+2(-1,-1,0,6,7,61)) \div 5 = (1,0,0,1,3,30)$. Then display the results, using 53 counting rods:

𝍠	𝍡	𝍠		𝍠
𝍠	𝍤			
	𝍢	𝍣		
𝍥	𝍨	**𝍢**		𝍠
𝍦	𝍣			𝍢
𝍮𝍠	𝍠𝍩𝍡	𝍣		𝍫

The augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ 1 & 0 & 4 & -3 & 0 & 4 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

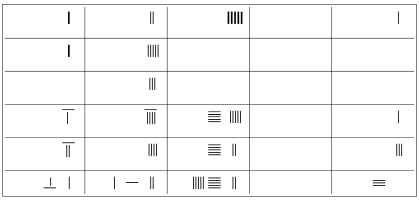
STEP 8. Subtract four times the fourth column from three times the third column, $3(1,0,4,-3,0,4)-4(2,5,3,9,4,112)=(-5,-20,0,-45,-16,-436)$, then change signs, and display the results, using 69 counting rods:



Our augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ 5 & 20 & 0 & 45 & 16 & 436 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 9. Next, subtract twenty times the fifth column from the third column, and then divide the result by three, $((5,20,0,45,16,436)+20(-1,-1,0,6,7,61))\div 3=(-5,0,0,55,52,552)$. Display the result on the counting board, using 75 counting rods:



The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ -5 & 0 & 0 & 55 & 52 & 552 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 10. Next eliminate another entry from the third column by adding five times the first column, $(-5,0,0,55,52,552)+5(1,0,0,1,3,30) = (0,0,0,60,67,702)$. Display the results, using 53 counting rods:

𝍠	𝍡			𝍠
𝍠	𝍤			
	𝍢			
𝍥	𝍨	𝍮		𝍠
𝍦	𝍣	𝍮𝍦		𝍢
𝍮𝍠	𝍠𝍩𝍡	𝍦 𝍡		𝍫

The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 11. Next, place the second column, using 77 counting rods:

𝍠	𝍡		𝍦	𝍠
𝍠	𝍤		𝍥	
	𝍢		𝍣	
𝍥	𝍨	𝍮	𝍤	𝍠
𝍦	𝍣	𝍮𝍦	𝍢	𝍢
𝍮𝍠	𝍠𝍩𝍡	𝍦 𝍡	𝍠𝍪𝍧	𝍫

Our augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ 7 & 6 & 4 & 5 & 3 & 128 \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

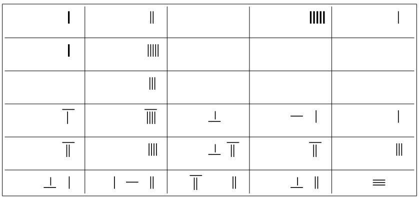
STEP 12. Then subtract four times the fourth column from three times the second column, $3(7,6,4,5,3,128)-4(2,5,3,9,4,112) = (13,-2,0,-21,-7,-64)$, change signs, and display the result, using 71 counting rods:

𝍠	𝍡		𝍩𝍢	𝍠
𝍠	𝍤		𝍡	
	𝍢			
𝍥	𝍨	𝍮	𝍪𝍠	𝍠
𝍦	𝍣	𝍮𝍦	𝍦	𝍢
𝍮𝍠	𝍠𝍩𝍡	𝍦 𝍡	𝍮𝍣	𝍫

The corresponding augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ -13 & 2 & 0 & 21 & 7 & 64 \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

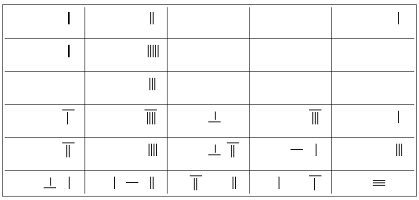
STEP 13. Add two times the fifth column to the second column and divide the result by three, $((-13,2,0,21,7,64)+2(-1,-1,0,6,7,61)) \div 3 = (-5,0,0,11,7,62)$, then display the result, using 67 counting rods:



On the augmented matrix, the result is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ -5 & 0 & 0 & 11 & 7 & 62 \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

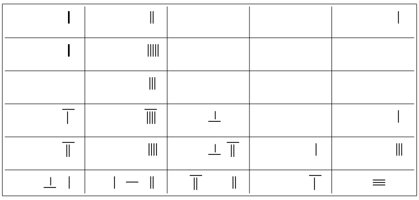
STEP 14. Add five times the first column to the second column and divide the results by 2, $((-5,0,0,11,7,62)+5(1,0,0,1,3,30)) \div 2 = (0,0,0,8,11,106)$, then display the results, using 62 counting rods:



Our augmented matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & 8 & 11 & 106 \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

STEP 15. Subtract four times the third column from thirty times the second column and divide the result by sixty two, $(30(0,0,0,8,11,106) - 4(0,0,0,60,67,702)) \div 62 = (0,0,0,0,1,6)$, and display the results, using 56 counting rods:



On the augmented matrix, the result is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 60 & 67 & 702 \\ 2 & 5 & 3 & 9 & 4 & 112 \\ -1 & -1 & 0 & 6 & 7 & 61 \end{bmatrix}$$

Back Substitution in the "Old Method"

We are now ready to find the solution by back substitution. For back substitution, the commentary apparently simply uses the intuitive approach familiar from modern mathematics. In the reconstruction above, the fifth unknown is given by the second column, namely, $x_5 = 6$. We can then substitute this value for the fifth unknown back into the third column to solve for the fourth unknown, namely, $x_4 = (702 - 6 \times 67) \div 60 = 5$, and so on.

Features of the "Old Method"

We can now summarize several points about the approach presented in the commentary attributed to Liu Hui. First, the approach taken here can perhaps best be described as serendipitous. On the one hand, no algorithm is presented, no method is described, and no rationale is provided for the seemingly fortuitous sequence of operations. The approach taken reduces the number of counting rods employed to what appears to be the minimum number required to solve the problem, namely, 77 rods. In this sense, the "old method" presented here suggests considerable expertise with this *single* problem. On the other hand, it lacks the computational advantages of the *fangcheng* procedure, namely, the generality, speed, efficiency, and lack of effort of the rote application of counting rod operations following simple patterns, along with the virtual assurance that fractions will not emerge in the process of back substitution. The commentary itself provides considerable evidence that suggests that the writer understood only the basics of *fangcheng* practice: (1) The textual record of the "old method" preserved in the commentary is hopelessly corrupt. That these recorded steps are this vague suggests that the writer might have known the result

—that problem 18 could be solved using 77 counting rods—but not the steps to reach that result. (2) There is no criticism or discussion of any of the specific features of the *fangcheng* procedure. (3) In particular, there is no criticism or discussion of the counterintuitive approach to back substitution, or even recognition that it is necessary to avoid the emergence of fractions. (4) The commentary to this problem, which is over 1600 characters,[37] fails to mention any of the difficulties that would be encountered in solving general *fangcheng* problems, such as systems with no solutions or indeterminate systems. (5) The commentary reduces columns using term-by-term division, for example, "the remainder can be halved" (lines 8 and 9 on page 316). Reducing columns by dividing by a common divisor of the entries results in a higher likelihood that fractions will emerge in the process of back substitution. (6) Finally, there is no mention of determinantal-style solutions to problems (this will be discussed in the following section).

Determinantal Solutions Recorded in Chinese Treatises

In addition to the more general solution using what is now called Gaussian elimination, determinantal-style calculations and solutions were also known for a distinctive class of *fangcheng* problems.[38] This distinctive class, in modern mathematical terms, is characterized by augmented matrix (8), as is explained below. Problem 13 of chapter 8 of the *Nine Chapters*, which I will refer to as the "well problem,"[39] is an exemplar for this distinctive class of *fangcheng* problems, and it is in commentaries to the *Nine Chapters* that records of determinantal calculations and solutions are preserved.

This section will demonstrate the following: (i) that determinantal calculations were used to find one of the unknowns in solving this distinctive class of problems; (ii) that this class of problems was at the time recognized as a distinct category; and (iii) that determinantal solutions were also known. It is this distinctive class of problems and solutions that can be found in Leonardo Pisano's writings, as will be shown in the section following this one.

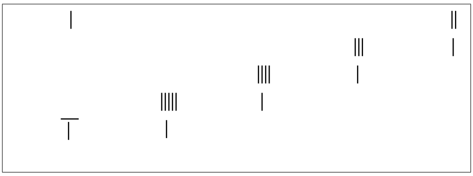
[37] Noted in Shen, Lun, and Crossley, *Nine Chapters*, 395.

[38] I thank John Crossley for suggesting the term "determinantal-style," which I will sometimes for convenience simply call "determinantal." "Determinantal-style calculations" refers to calculations made in a manner we might now call determinantal; "determinantal-style solutions" refers to solutions for all the unknowns calculated in a manner we might now call determinantal.

[39] See footnote 8 on page 290.

Solutions to the Well Problem on the Counting Board

The well problem, when displayed on the counting board, is as follows:



$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & y \\ 0 & 3 & 1 & 0 & 0 & y \\ 0 & 0 & 4 & 1 & 0 & y \\ 0 & 0 & 0 & 5 & 1 & y \\ 1 & 0 & 0 & 0 & 6 & y \end{bmatrix}. \tag{7}$$

Using modern algebraic notation, we can also write the well problem as a linear system of five equations in six unknowns ($x_1, x_2, \dots, x_5$, and y) as follows,

$$2x_1 + x_2 = y, \quad 3x_2 + x_3 = y, \quad 4x_3 + x_4 = y, \quad 5x_4 + x_5 = y, \quad 6x_5 + x_1 = y.$$

Because the well problem is a system of five conditions in six unknowns, the solutions are not unique. It is not, however, "indeterminate" in the modern sense. In the earliest received version of the *Nine Chapters*, the $(n+1)^{\text{th}}$ unknown is simply stipulated to be 721, with no explanation provided. A later commentary by Jia Xian 賈憲 (fl. 1023–1063), preserved in Yang Hui's 楊輝 (c. 1238–c. 1298) *Nine Chapters on the Mathematical Arts, with Detailed Explanations* (*Xiang jie jiuzhang suanfa* 詳解九章算法, c. 1261),[40] explains how the $(n+1)^{\text{th}}$ unknown is found. The diagonal terms are multiplied together, the remaining terms are multiplied together (that is, the super-diagonal and the term in the corner), and then the two products are added, yielding

$$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 721.$$

Setting the $(n+1)^{\text{th}}$ unknown $y = 721$ gives a linear system of five conditions in five unknowns,

$$2x_1 + x_2 = 721,\ 3x_2 + x_3 = 721,\ 4x_3 + x_4 = 721,\ 5x_4 + x_5 = 721,\ 6x_5 + x_1 = 721,$$

which then has a unique solution, which can be found by Gaussian elimination,

$$x_1 = 265, \quad x_2 = 191, \quad x_3 = 148, \quad x_4 = 129, \quad x_5 = 76.$$

Jia Xian's commentary further explains that problems similar to the well problem form a category, and explains their solution. Written in modern mathematical terms, the well problem serves as an exemplar for systems of n conditions in $n+1$ unknowns of the following distinctive form:

$$\begin{bmatrix} k_1 & l_1 & 0 & \cdots & 0 & y \\ 0 & k_2 & l_2 & \ddots & \vdots & y \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & k_{n-1} & l_{n-1} & y \\ l_n & 0 & \cdots & 0 & k_n & y \end{bmatrix}. \tag{8}$$

40 Yang Hui, *Xiang jie jiuzhang suanfa* 詳解九章算法 [Nine chapters on the mathematical arts, with detailed explanations], in *ZKJDT*.

To solve problems of this distinctive class, the $(n+1)^{\text{th}}$ unknown is first assigned a value as follows. In modern terminology, the problem is transformed into n conditions in n unknowns by setting the $(n+1)^{\text{th}}$ unknown $y = \det A$, where A is the matrix of coefficients. That is,

$$y = \det A = \begin{vmatrix} k_1 & l_1 & 0 & \cdots & 0 \\ 0 & k_2 & l_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & k_{n-1} & l_{n-1} \\ l_n & 0 & \cdots & 0 & k_n \end{vmatrix} = k_1 k_2 k_3 \ldots k_n \pm l_1 l_2 l_3 \ldots l_n,$$

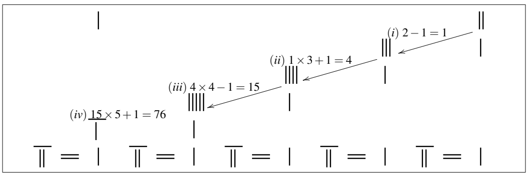
where here, and throughout this article, $\pm$ is $+$ if n is odd and $-$ if n is even.

It should also be noted that this calculation apparently served as a "determinant" in the modern sense: the linear system of five conditions in five unknowns will have a unique solution if and only if $y = \det A \neq 0$, that is, in the well problem, if and only if the "depth of the well" (*jing shen* 井深) is not zero.

Problems of this distinctive class, exemplified by augmented matrix (8), also have determinantal-style solutions. For example, if, as in the well problem, we have five conditions in six unknowns, $l_i = 1$ for $1 \leq i \leq 5$, and we set the $(n+1)^{\text{th}}$ unknown $y = \det A$, the solutions for the remaining unknowns $x_1, x_2, \ldots, x_5$ are as follows:

$$x_1 = (((k_2 - 1)k_3 + 1)k_4 - 1)k_5 + 1, \tag{9}$$
$$x_2 = (((k_3 - 1)k_4 + 1)k_5 - 1)k_1 + 1, \tag{10}$$
$$x_3 = (((k_4 - 1)k_5 + 1)k_1 - 1)k_2 + 1, \tag{11}$$
$$x_4 = (((k_5 - 1)k_1 + 1)k_2 - 1)k_3 + 1, \tag{12}$$
$$x_5 = (((k_1 - 1)k_2 + 1)k_3 - 1)k_4 + 1. \tag{13}$$

These solutions could easily have been computed on a counting board, using what might be termed a determinantal calculation, as the following diagram for the calculation of the fifth unknown, x_5, in the well problem demonstrates:



The remaining unknowns are easily calculated following similar patterns.

The earliest record of such a determinantal solution that I have found in Chinese treatises is preserved in a commentary on the well problem in Fang Zhongtong's 方中通 (1634–1698) *Numbers and Measurement, An Amplification* (*Shu du yan* 數度衍, c. 1661).[41] Fang describes, using words to denote positions in the array of numbers,

[41] Fang Zhongtong, *Shu du yan* 數度衍 [Numbers and measurement, an amplification], in *SKQS*.

a solution for the fifth unknown of the well problem. This solution is valid for a slightly more general class of problems, which, written using modern terminology as an augmented matrix, is of the following form,

$$\begin{bmatrix} k_1 & l_1 & 0 & \cdots & 0 & b_1 \\ 0 & k_2 & l_2 & \ddots & \vdots & b_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & k_{n-1} & l_{n-1} & b_{n-1} \\ l_n & 0 & \cdots & 0 & k_n & b_n \end{bmatrix}. \tag{14}$$

The solution, as calculated by Fang, written in modern notation, is given by the following,

$$x_5 = \frac{(((k_1b_5 - l_5b_1)k_2 + l_5l_1b_2)k_3 - l_5l_1l_2b_3)k_4 + l_5l_1l_2l_3b_4}{k_1k_2k_3k_4k_5 + l_5l_1l_2l_3l_4}.$$

And if, beginning with augmented matrix (8), we set the $(n+1)^{\text{th}}$ unknown $y = \det A = k_1k_2k_3k_4k_5 + l_5l_1l_2l_3l_4$, we have a somewhat simpler solution for the fifth unknown,

$$x_5 = ((((k_1 - l_5)k_2 + l_5l_1)k_3 - l_5l_1l_2)k_4 + l_5l_1l_2l_3.$$

Again, the remaining unknowns can be calculated in a similar manner.

The solution given by Fang, it can be shown, can be generalized for any n ($n \geq 3$). For example, given a system of n conditions in n unknowns of the more general form of augmented matrix (14), the value of the n^{th} unknown is given by

$$x_n = \frac{\begin{array}{c}((\cdots((((k_1b_n - l_nb_1)k_2 + l_nl_1b_2)k_3 - l_nl_1l_2b_3)k_4 + l_nl_1l_2l_3b_4)\cdots) \\ \cdot k_{n-2} \mp l_nl_1l_2\cdots l_{n-3}b_{n-2})k_{n-1} \pm l_nl_1l_2\cdots l_{n-2}b_{n-1}\end{array}}{k_1k_2k_3\cdots k_{n-1}k_n \pm l_nl_1l_2l_3\cdots l_{n-1}}, \tag{15}$$

where again $\pm$ is $+$ for n odd and $-$ for n even. A similar solution can be found for each of the n unknowns.

This solution, although it looks complicated when written in modern mathematical notation, is not difficult to compute on the counting board by following simple patterns, as illustrated below. We set the terms $l_nl_1, -l_nl_1l_2, l_nl_1l_2l_3, \ldots, \mp l_nl_1\cdots l_{n-2}$ in the empty positions in the left-hand column. The numerator given in equation (15) is then computed by a series of cross-multiplications, working from the outermost corners inward. In the first step, the opposite corners of the array are multiplied together, and subtracted, giving the result $k_1 b_n - l_n b_1$:

$\mathbf{l_n}$	0	$\cdots$	0	0	$\mathbf{k_1}$
$-l_n l_1$	0	$\cdots$	0	k_2	l_1
$l_n l_1 l_2$	$\vdots$	$\iddots$	k_3	l_2	0
$\vdots$	0	$\iddots$	$\iddots$	0	0
$\mp l_n l_1 \cdots l_{n-2}$	k_{n-1}	l_{n-2}	$\iddots$	$\vdots$	$\vdots$
k_n	l_{n-1}	0	$\cdots$	0	0
$\mathbf{b_n}$	b_{n-1}	$\cdots$	b_3	b_2	$\mathbf{b_1}$

$$k_1 b_n - l_n b_1$$

In each remaining step, we follow the same pattern, moving one position inward, cross-multiplying, and then subtracting. We continue in this manner until we reach k_{n-1}, which is the final step:

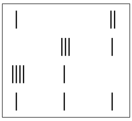
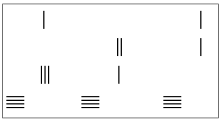
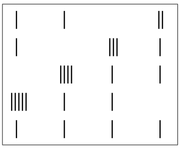
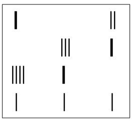
l_n	0	$\cdots$	0	0	k_1
$-l_n l_1$	0	$\cdots$	0	k_2	l_1
$l_n l_1 l_2$	$\vdots$	$\iddots$	k_3	l_2	0
$\vdots$	0	$\iddots$	$\iddots$	0	0
$\mathbf{\mp l_n l_1 \cdots l_{n-2}}$	$\mathbf{k_{n-1}}$	l_{n-2}	$\iddots$	$\vdots$	$\vdots$
k_n	l_{n-1}	0	$\cdots$	0	0
b_n	$\mathbf{b_{n-1}}$	$\cdots$	b_3	b_2	b_1

$$((\cdots((k_1 b_n - l_n b_1)k_2 + l_n l_1 b_2)\cdots)\cdot k_{n-2} \mp l_n l_1 l_2 \cdots l_{n-3} b_{n-2})$$
$$\cdot k_{n-1} \pm l_n l_1 l_2 \cdots l_{n-2} b_{n-1}$$

This then gives the numerator for (15).

Although the earliest extant record I have found of a determinantal solution is Fang's *Numbers and Measurement* from the seventeenth century, there is important evidence that some determinantal solutions, such as those given in equations (9)–(13), were known at the time of compilation of the *Nine Chapters*. Among the eighteen problems in "*Fangcheng*," chapter 8 of the *Nine Chapters*, there are four more problems, in addition to the well problem, that are variants of the distinctive form given by augmented matrices (8) and (14), for which the absence of entries permits relatively simple determinantal solutions, namely problems 3, 12, 14, and 15. For each of these problems, following simple patterns, it is possible to compute the values for all of the unknowns; and from the point of view of the original *fangcheng* procedure (a variant of Gaussian elimination) presented in the *Nine Chapters*, there is nothing noteworthy about any of these problems that would explain their inclusion in the text. A brief conspectus of these problems is presented in Table 2 on the facing page.

Table 2: From among the eighteen problems in "*Fangcheng*," chapter 8 of the *Nine Chapters*, here are four problems, in addition to the well problem given in equation (7), that are variants of augmented matrices (8) and (14), for which the absence of entries allows simple determinantal solutions.

Problem	*Counting Board*	*Augmented Matrix*	*Notes*
3		$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 4 & 1 \end{bmatrix}$	Variant with three conditions in three unknowns.
12		$\begin{bmatrix} 1 & 1 & 0 & 40 \\ 0 & 2 & 1 & 40 \\ 1 & 0 & 3 & 40 \end{bmatrix}$	Variant with three conditions in three unknowns.
14		$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 1 \\ 1 & 1 & 0 & 5 & 1 \end{bmatrix}$	Variant with four conditions in four unknowns.
15		$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 1 \\ -1 & 0 & 4 & 1 \end{bmatrix}$	Variant with three conditions in three unknowns, with negative coefficients.

Later Chinese mathematical treatises continue to record numerous problems that are variants of the distinctive form given in augmented matrices (8) and (14): often as many as 10–20% of the problems are similar. It was because of the importance of these problems in Chinese mathematical treatises that I chose an example of a *fangcheng* problem with nine conditions in nine unknowns of the form of augmented matrix (14), shown here in Figure 1 on the next page, for the cover of *Chinese Roots of Linear Algebra*.

Problems recorded in Leonardo Pisano's *Liber Abaci*

Problems of the form of augmented matrix (8), for which the well problem was an exemplar, are so distinctive that they can serve as a "fingerprint," a kind of unique identifier. In my research for *Chinese Roots of Linear Algebra*, I searched through

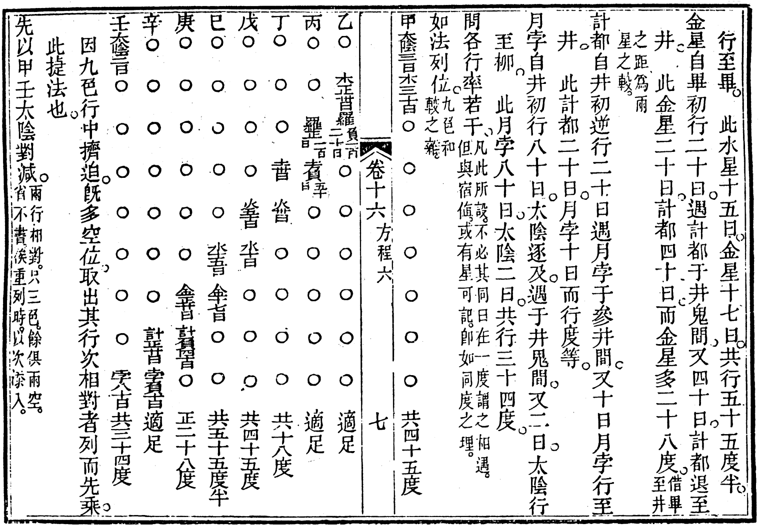


Figure 1: An array of the form given in augmented matrix (14), representing a problem with nine conditions in nine unknowns, from Mei Wending's 梅文鼎 (1633–1721) *Fangcheng lun* 方程論 (On *fangcheng*, c. 1674, photolithographic reprint from the Mei Juecheng Chengxuetang 梅㲄成承學堂 printing of the *Mei shi congshu ji yao* 梅氏叢書輯要).

thousands of matrices recorded in modern mathematical treatises,[42] and found only one example of a problem that was similar (a problem with three equations in three unknowns). Because these problems are so distinctive, I then searched for similar problems in European mathematical treatises from the seventeenth and eighteenth centuries, including the extant mathematical writings of Gauss and Leibniz. Unable to find similar problems, I felt that the most that I could definitively conclude is that "[t]he essentials of the methods used today … in modern linear algebra *were not first discovered* by Leibniz or by Gauss."[43] It turns out that I was looking for problems of the form given in augmented matrix (8) in seventeenth- and eighteenth-century European texts, at least four centuries after they were transmitted to Europe.

[42] Strang, *Linear Algebra and Its Applications*; Denis Serre, *Matrices: Theory and Applications* (New York: Springer, 2002); Fuzhen Zhang, *Matrix Theory: Basic Results and Techniques* (New York: Springer, 1999); Stewart, *Matrix Algorithms*; Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis* (New York: Cambridge University Press, 1991); Charles G. Cullen, *Matrices and Linear Transformations*, 2nd ed. (New York: Dover, 1990); Roger A. Horn and Charles R. Johnson, *Matrix Analysis* (New York: Cambridge University Press, 1985); G. W. Stewart, *Introduction to Matrix Computations* (New York: Academic Press, 1973); Kenneth Hoffman and Ray Alden Kunze, *Linear Algebra*, 2nd ed. (Englewood Cliffs, NJ: Prentice-Hall, 1971); Thomas Muir, *The Theory of Determinants in the Historical Order of Development*, 4 vols. (London: Macmillan, 1906–1923).

[43] Hart, *Chinese Roots of Linear Algebra*, p. 192, emphasis in italics added.

Many examples of problems equivalent to the form given in augmented matrix (8) can be found in the work of Leonardo Pisano (c. 1170–c. 1250), more commonly known today by the name Fibonacci. Not only are these problems themselves distinctive, the solutions to these problems are even more so. The solutions to these problems are without analog in modern mathematics, and in fact these solutions are esoteric enough that they have not been adequately analyzed in previous studies of Fibonacci's mathematics.[44] To put it another way, it was because of my familiarity with Chinese *fangcheng* problems and their solutions that the problems and solutions I found in Fibonacci made sense.

Here I will briefly examine one problem of the form of augmented matrix (8), "Another Problem on Five Men," and its solution (as Table 3 on page 341 shows, this is but one among several examples). In Fibonacci, this problem is recorded as a word problem. In modern mathematical terms, it is a system of five conditions in six unknowns, which can be written as follows:

$$x_1 + \frac{2}{3}x_2 = y,\ x_2 + \frac{4}{7}x_3 = y,\ x_3 + \frac{5}{11}x_4 = y,\ x_4 + \frac{6}{13}x_5 = y,\ x_5 + \frac{8}{19}x_1 = y. \quad (16)$$

If we write this as an augmented matrix, to facilitate comparison with the *fangcheng* problems, we can immediately see that it is similar in form to that given by augmented matrix (8):

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 & 0 & 0 & y \\ 0 & 1 & \frac{4}{7} & 0 & 0 & y \\ 0 & 0 & 1 & \frac{5}{11} & 0 & y \\ 0 & 0 & 0 & 1 & \frac{6}{13} & y \\ \frac{8}{19} & 0 & 0 & 0 & 1 & y \end{bmatrix}. \quad (17)$$

To solve the problem, Fibonacci writes the problem out as a series of fractions arranged contiguously, as follows,

> The details of the taking are written in order thus: $\frac{8}{19}\frac{6}{13}\frac{5}{11}\frac{4}{7}\frac{2}{3}$.[45]

The operations he uses to solve the problem are not those familiar from modern mathematics. First, the $(n+1)^{\text{th}}$ unknown is found as follows:

[44] Johannes Tropfke et al., *Geschichte der Elementarmathematik, Band 1, Arithmetik und Algebra*, 4th ed., 3 vols. (New York: Walter de Gruyter, 1980); Jacques Sesiano, "The Appearance of Negative Solutions in Mediaeval Mathematics," *Archive for History of Exact Sciences* 32 (1985): 105–50; Maryvonne Spiesser, "Problèmes linéaires dans le Compendy de la praticque des nombres de Barthélemy de Romans et Mathieu Préhoude (1471): Une approche nouvelle basée sur des sources proches du Liber abbaci de Léonard de Pise," *Historia Mathematica* 27 (2000): 362–83; John Hannah, "False Position in Leonardo of Pisa's Liber Abbaci," *Historia Mathematica* 34 (2007): 306–32; idem, "Conventions for Recreational Problems in Fibonacci's Liber Abbaci," *Archive for History of Exact Sciences* 65 (2010): 155–80.

[45] Laurence E. Sigler, *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation* (New York: Springer, 2002), 345; *Scritti di Leonardo Pisano matematico del secolo decimoterzo. I. Il Liber abbaci di Leonardo Pisano*, ed. Baldassare Boncompagni (Roma: Tipografia delle Scienze Matematiche e Fisiche, 1857), 234.

> And all of the numbers that are under the fraction are multiplied together; there will be 57057. *As the number of men is odd*, the product of the numbers that are over the fractions is *added* to this, that is, the 2 times the 4 times the 5 times the 6 [times the 8]; there will be 58977, which is had for the price of the horse.[46]

The above calculation notes that when "the number of men is odd" the products of the two sets of numbers (the numerators and the denominators) are to be added. That is, the price of the horse is given by

$$3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 + 2 \cdot 4 \cdot 5 \cdot 6 \cdot 8 = 58977. \tag{18}$$

The problem is then solved for each unknown in succession. The calculation for the first unknown is explained in Fibonacci as follows:

> And as the first man's bezants are had, the upper number of the fraction of the takings is subtracted from the lower number of the same fraction, that is the 2 from the 3; there remains 1, and it is multiplied by the 7; there will be 7, to which you add the product of the 2 and the 4; there will be 15, which you multiply by the 11; there will be 165, from which you subtract the product of the 2 and the 4 and the 5; there remains 125, which you multiply by the 13; there will be 1625, to which you add the product of the 2 and the 4 and the 5 and the 6; there will be 1865, which is multiplied by the 19; there will be 35435, and the first man has this many.[47]

Written out using modern mathematical notation, this is the following calculation,

$$((((3-2) \cdot 7 + 2 \cdot 4) \cdot 11 - 2 \cdot 4 \cdot 5) \cdot 13 + 2 \cdot 4 \cdot 5 \cdot 6) \cdot 19 = 35435. \tag{19}$$

Each of the remaining unknowns is calculated in a similar manner.

More abstractly, in modern mathematical terms, "Another Problem on Five Men" is an exemplar for the following problem with 5 conditions in 6 unknowns:

$$x_1 + \frac{k_1}{l_1}x_2 = y,\ x_2 + \frac{k_2}{l_2}x_3 = y,\ x_3 + \frac{k_3}{l_3}x_4 = y,\ x_4 + \frac{k_4}{l_4}x_5 = y,\ x_5 + \frac{k_5}{l_5}x_1 = y. \tag{20}$$

Written as an augmented matrix (again, to facilitate comparison with the *fangcheng* problems), we have the following:

$$\begin{bmatrix} 1 & \frac{k_1}{l_1} & 0 & 0 & 0 & y \\ 0 & 1 & \frac{k_2}{l_2} & 0 & 0 & y \\ 0 & 0 & 1 & \frac{k_3}{l_3} & 0 & y \\ 0 & 0 & 0 & 1 & \frac{k_4}{l_4} & y \\ \frac{k_5}{l_5} & 0 & 0 & 0 & 1 & y \end{bmatrix}. \tag{21}$$

[46] Sigler, *Fibonacci's Liber Abaci*, 345, emphasis in italics added; Boncompagni, *Liber abbaci*, 234, interpolation based on the mathematics, added.

[47] Sigler, *Fibonacci's Liber Abaci*, 345; Boncompagni, *Liber abbaci*, 234–35.

If we follow the steps given in the word problem, the first step is to write the problem as a series of fractions arranged contiguously in the following manner:

$$\frac{k_5}{l_5}\frac{k_4}{l_4}\frac{k_3}{l_3}\frac{k_2}{l_2}\frac{k_1}{l_1}. \tag{22}$$

Then we calculate the $(n+1)^{\text{th}}$ unknown, y, by setting it to the value

$$k_1k_2k_3k_4k_5 \pm l_1l_2l_3l_4l_5, \tag{23}$$

which is the numerator of the determinant of the coefficient matrix,

$$\det A = \begin{vmatrix} 1 & \frac{k_1}{l_1} & 0 & 0 & 0 \\ 0 & 1 & \frac{k_2}{l_2} & 0 & 0 \\ 0 & 0 & 1 & \frac{k_3}{l_3} & 0 \\ 0 & 0 & 0 & 1 & \frac{k_4}{l_4} \\ \frac{k_5}{l_5} & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{k_1k_2k_3k_4k_5 \pm l_1l_2l_3l_4l_5}{l_1l_2l_3l_4l_5}. \tag{24}$$

The solutions, again written in modern mathematical notation, are then

$$x_1 = ((((l_1 - k_1)l_2 + k_1k_2)l_3 - k_1k_2k_3)l_4 + k_1k_2k_3k_4)l_5, \tag{25}$$
$$x_2 = ((((l_2 - k_2)l_3 + k_2k_3)l_4 - k_2k_3k_4)l_5 + k_2k_3k_4k_5)l_1, \tag{26}$$
$$x_3 = ((((l_3 - k_3)l_4 + k_3k_4)l_5 - k_3k_4k_5)l_1 + k_3k_4k_5k_1)l_2, \tag{27}$$
$$x_4 = ((((l_4 - k_4)l_5 + k_4k_5)l_1 - k_4k_5k_1)l_2 + k_4k_5k_1k_2)l_3, \tag{28}$$
$$x_5 = ((((l_5 - k_5)l_1 + k_5k_1)l_2 - k_5k_1k_2)l_3 + k_5k_1k_2k_3)l_4. \tag{29}$$

As with *fangcheng* problems, these problems were solved using simple patterns in two dimensions. I will briefly summarize these calculations below.[48]

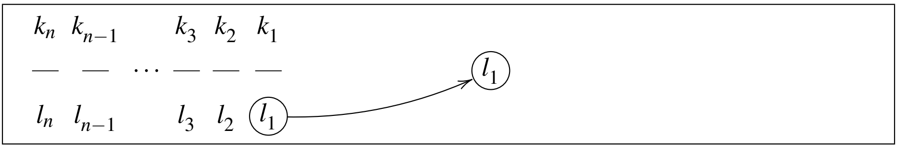
STEP 0. In the margins of Leonardo's *Liber Abaci*, in the edition preserved by Boncompagni, there are diagrams of the fractions arranged contiguously, which can be written in general terms using modern notation as follows:

$$\boxed{\begin{array}{ccccccc} k_n & k_{n-1} & & k_3 & k_2 & k_1 \\ — & — & \cdots & — & — & — \\ l_n & l_{n-1} & & l_3 & l_2 & l_1 \end{array}}$$

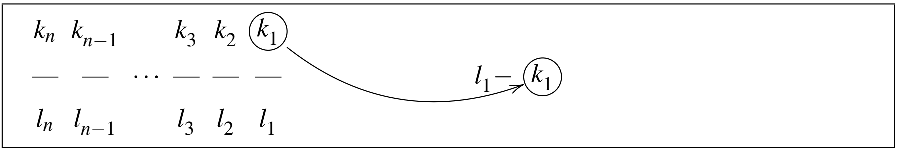
STEP 1. The first step is simply to begin with the lower term from the first fraction. (In this reconstruction, I have placed the calculations to the right of the contiguously arranged fractions. In the text, there is no indication of where or even whether

48 These results are the summary of research I am preparing for publication.

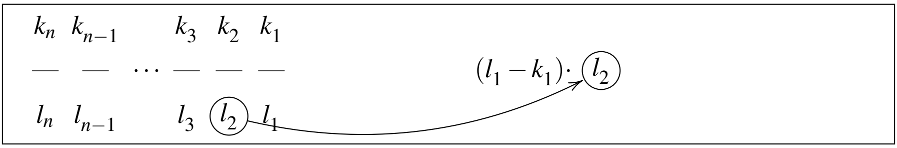
successive results were recorded.)



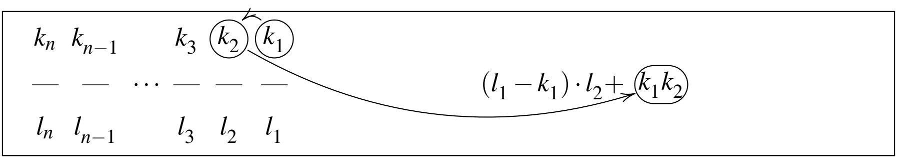
STEP 2. Second, we subtract the upper term of the first fraction from the lower term of the first fraction, yielding $l_1 - k_1$:



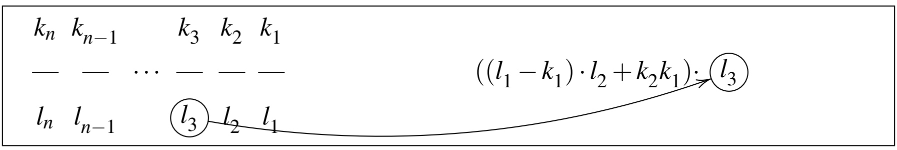
STEP 3. The next step is to multiply the results of the last operation, $l_1 - k_1$, by the lower term of the second fraction, l_2:



STEP 4. In the following step, we multiply the upper terms of the first two fractions together, giving $k_1 k_2$. In the steps that follow, we will alternate between subtracting and adding terms $k_1 \cdots k_i$ to the previous result. In step 2 above, we subtracted k_1, so here we add $k_1 k_2$ to the previous result, $(l_1 - k_1) \cdot l_2$, giving $(l_1 - k_1) \cdot l_2 + k_1 k_2$:



STEP 5. We then multiply the results of the previous step, $((l_1 - k_1) \cdot l_2 + k_1 k_2)$, by the lower term of the third fraction, l_3:



STEP 6. We multiply together, from right to left, the upper terms of the first three fractions, giving $k_1k_2k_3$. We then subtract this from the previous result $((l_1-k_1)\cdot l_2 + k_1k_2)\cdot l_3$, giving $((l_1-k_1)\cdot l_2+k_1k_2)\cdot l_3-k_1k_2k_3$:

$$\frac{k_n}{l_n} \quad \frac{k_{n-1}}{l_{n-1}} \quad \cdots \quad \frac{k_3}{l_3} \quad \frac{k_2}{l_2} \quad \frac{k_1}{l_1} \qquad ((l_1-k_1)\cdot l_2+k_1k_2)\cdot l_3 - k_1k_2k_3$$

$$\vdots$$

STEP $(2n-2)$. We continue to follow these simple patterns. The next-to-last step is to multiply the upper terms of all but the last of the fractions, from right to left, giving $k_1k_2\cdots k_{n-1}$. Since we alternate between adding and subtracting these terms, we will add this term if n is odd, and subtract if n is even, as follows:

$$\frac{k_n}{l_n} \quad \frac{k_{n-1}}{l_{n-1}} \quad \cdots \quad \frac{k_3}{l_3} \quad \frac{k_2}{l_2} \quad \frac{k_1}{l_1} \qquad (\cdots((l_1-k_1)\cdot l_2+k_1k_2)\cdots)\cdot l_{n-1} \pm k_1k_2\cdots k_{n-1}$$

STEP $(2n-1)$. The final step is as follows:

$$\frac{k_n}{l_n} \quad \frac{k_{n-1}}{l_{n-1}} \quad \cdots \quad \frac{k_3}{l_3} \quad \frac{k_2}{l_2} \quad \frac{k_1}{l_1} \qquad ((\cdots((l_1-k_1)\cdot l_2+k_1k_2)\cdots)\cdot l_{n-1} \pm k_1k_2\cdots k_{n-1})\cdot l_n$$

This is only one example of several similar problems recorded in Fibonacci's *Liber Abaci*, as shown in Table 3.

Table 3: Several examples from Fibonacci's *Liber Abaci* of problems of the form given in augmented matrix (8) (the last problem listed here, "On Five Men Who Bought a Horse," is solved in *Liber Abaci* by a variant of false position).

Title of problem	*Problem written as an augmented matrix*
"On the Purchase of a Horse by Three Men, When Each One Takes Some Bezants from the Others." Sigler, *Fibonacci's Liber Abaci*, 338–39.	$\begin{bmatrix} 1 & \frac{1}{3} & 0 & y \\ 0 & 1 & \frac{1}{4} & y \\ \frac{1}{5} & 0 & 1 & y \end{bmatrix}$

Continued on the next page

Title of problem	*Augmented Matrix*
"Another Problem on Three Men According to the Abovewritten Method." Sigler, *Fibonacci's Liber Abaci*, 341.	$\begin{bmatrix} 1 & \frac{2}{3} & 0 & y \\ 0 & 1 & \frac{4}{7} & y \\ \frac{5}{9} & 0 & 1 & y \end{bmatrix}$
"On the Same with Four Men." Ibid., 341–42.	$\begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & y \\ 0 & 1 & \frac{1}{4} & 0 & y \\ 0 & 0 & 1 & \frac{1}{5} & y \\ \frac{1}{6} & 0 & 0 & 1 & y \end{bmatrix}$
"On Five Men Who Bought a Horse." Ibid., 458.	$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & y \\ 0 & 1 & \frac{1}{3} & 0 & 0 & y \\ 0 & 0 & 1 & \frac{1}{4} & 0 & y \\ 0 & 0 & 0 & 1 & \frac{1}{5} & y \\ \frac{1}{6} & 0 & 0 & 0 & 1 & y \end{bmatrix}$

From this analysis of Leonardo's *Liber Abaci*, we can reach the following conclusions:

1. Problems of the distinct form of augmented matrices (8) and (14), with solutions that are quite esoteric, recorded in Chinese treatises dating from about the first century CE, are also recorded in Leonardo's *Liber Abaci*.
2. The solutions are valid for any number of unknowns, and the same simple visual patterns work for all of the unknowns.
3. Leonardo records these solutions in such detail that there can be no question as to how they were solved, and the mathematical practice can be reliably reconstructed.
4. Leonardo does not, however, provide an explanation of the actual mathematical practice, but only records the calculations.
5. These methods do not require literacy, and in fact, the translation of this two-dimensional mathematical practice into narrative renders it almost incomprehensible.
6. Calculations of this form were quite common in this period. More specifically, these calculations are similar in form to what is today known as Horner's method.[49]

[49] I thank John Crossley for this important suggestion noting the similarity. For a detailed explanation of this method of finding roots, see Martzloff, *History of Chinese Mathematics*, "The Extraction of Roots," pp. 221–49. This method can be found, in various forms, in the *Nine Chapters* and later Chinese mathematical treatises. Polynomial equations that are today written in the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$$

7. Although these problems are preserved in Leonardo's *Liber Abaci*, to my knowledge they were not recorded in other European treatises, perhaps because the solution is so esoteric.

The *Guide to Calculation*

Having traced the history of *fangcheng* problems, we are now in a better position to analyze the *Guide to Calculation* attributed to Li and Ricci, together with Xu Guangqi's preface, in which he claims that Western mathematics is in every way superior. Most of the *Guide to Calculation* is indeed translated from Clavius's *Epitome*, and most of it consists of elementary mathematics, beginning with addition, subtraction, multiplication, and division. The most "modern" mathematics in the *Guide to Calculation* is in chapters in the second volume (*tong bian* 通編) on what we now call linear algebra. We will first examine chapter 4 of the second volume, which presents problems from Clavius's *Epitome* with 2 conditions in 2 unknowns, and then explicitly compares them to similar problems in Chinese treatises, in order to pronounce the Western mathematics superior. Next, we will examine chapter 5. There are no problems with *n* conditions in *n* unknowns ($n \geq 3$) in Clavius's *Epitome*, so *fangcheng* problems from Chinese sources were simply purloined without attribution. There is little evidence that Li, Xu, Yang, and their collaborators understood these problems: they offer no analysis, criticisms, or alternative methods. Next, we will examine Xu's preface, and in particular his claim that Western mathematics was in every way superior.

Chapter 4, "Adding and borrowing for mutual proof"

The fourth chapter (*juan si* 卷四), titled "Adding and borrowing for mutual proof" (*die jie hu zheng* 疊借互徵), is a translation of chapter 23 of Clavius's *Epitome*, titled "Regula falsi duplicis positionis" (the rule of double false position). These problems, which likely circulated throughout Eurasia, are equivalent to 2 conditions in 2 unknowns, and are similar to excess-deficit problems commonly recorded in Chinese mathematical treatises of the period. The *Guide to Calculation* employs two main strategies to assert the superiority of Western mathematics:

were solved by calculating in the following manner,

$$(((\cdots(((a_n x + a_{n-1})x + a_{n-2})x + a_{n-3})\cdots)x + a_2)x + a_1)x + a_0.$$

In the case that $a_n = 1$, this reduces to

$$(((\cdots(((x + a_{n-1})x + a_{n-2})x + a_{n-3})\cdots)x + a_2)x + a_1)x + a_0.$$

This method is more efficient than the modern method. See Donald Ervin Knuth, *The Art of Computer Programming*, 2nd ed. (Reading, MA: Addison-Wesley, 1981), II, 467, cited in Martzloff, *History of Chinese Mathematics*, 246.

1. First, the problems from Clavius's *Epitome* are alleged to predate Chinese problems:

   > Old [Chinese mathematical treatises] had a section "excess and deficit,"[50] on the whole of the same category as this; but this [the contents of this chapter of the *Guide to Calculation*] predates [the time] when [the Chinese methods of] "excess and deficit" did not yet exist.
   >
   > 舊有盈朒一章。大都類此。而此則於未有盈朒之先。

2. Second, the *Guide to Calculation* imputes superiority of methods from the *Epitome* over Chinese methods by a purported comparison. Problems from Chinese sources are appended to the end of the chapter, clearly noting that they are copied: before the problems, "The old method 'excess and deficit' chapter is one that people are constantly studying, so [we have] appended several problems below for comparison" 舊法盈朒章人所恆習亦附數條于後相比擬; after the problems, "the previous are numerator-denominator excess-deficit" 右母子盈朒.

It should be noted that there is no evidence to support either of these claims of superiority.

Chapter 5, "Method for addition, subtraction, and multiplication of heterogeneous [elements]"

Clavius' *Epitome* does not record any problems equivalent to *fangcheng* 方程, that is, problems equivalent to the solution of systems of n conditions in n unknowns ($n \geq 3$) in modern linear algebra. In general, such problems were rarely recorded in European treatises of the period; rather, the vast majority of written records of these problems are from Chinese treatises.

The problems in chapter 5, "Method for addition, subtraction, and multiplication of heterogeneous [elements]" [*za he jiao cheng fa* 雜和較乘法], are copied, without attribution, problem for problem, from Chinese sources. Many of these problems are copied from Cheng Dawei's 程大位 (1533–1606) *Comprehensive Source of Mathematical Methods* (*Suanfa tongzong* 算法统宗): problem twelve is Cheng's problem three; problem fourteen is Cheng's problem five; problem sixteen is Cheng's problem seven; and problem seventeen is Cheng's problem eight. Changes are purely cosmetic: the order of sentences in the problems is changed, names of variables are changed, but none of the numbers used in the calculation is changed. All the solutions are correct, but no additional analysis or explanation is offered by the Jesuits' Chinese collaborators.

[50] The title used in *Jiuzhang suanshu* 九章算術 [Nine chapters on the mathematical arts], in *ZKJDT*, is "Excess and deficit" (*ying bu zu*). Wu Jing 吳敬 (fl. 1450), *Jiuzhang xiang zhu bilei suanfa da quan* 九章詳註比類算法大全 [Complete collection of the mathematical arts of the nine chapters, with detailed commentary, arranged by category], in *ZKJDT*, uses *ying bu zu*, but elsewhere uses *ying fei* 盈朏. Cheng Dawei 程大位 (1533–1606), *Suanfa tongzong* 算法统宗 [Comprehensive source of mathematical methods], in *ZKJDT*, uses "ying nü" 盈朒.

In particular, the well problem is copied, without any attribution or acknowledgment, from Chinese mathematical treatises. Again, Clavius's *Epitome* includes no similar problems. The original source of the problem is not certain: Cheng's *Comprehensive Source of Mathematical Methods* does not contain the well problem; the most likely source is Wu Jing's 吳敬 (fl. 1450) *Complete Collection of the Mathematical Arts of the Nine Chapters, with Detailed Commentary, Arranged by Category* (*Jiuzhang xiang zhu bilei suanfa da quan* 九章詳註比類算法大全).

Below is a translation of this problem, as found in the *Guide to Calculation*.

> Problem: The depth of a well is unknown. Using 2 of A's ropes the water is not reached, borrowing 1 of B's ropes to supplement it, the water is reached. Using 3 of B's ropes then borrowing 1 of C's, using 4 of C's ropes then borrowing 1 of D's, using 5 of D's ropes then borrowing 1 of E's, using 6 lengths of E's ropes, then borrowing 1 of A's, then all reach the water. How much is the depth of the well? How much is [the length of] each rope?
>
> 問井不知深。用甲繩二不及泉，借乙繩一補之，及泉。用乙繩三則借丙一，用丙繩四則借丁一，用丁繩五則借戊一，用戊繩六條借甲一，乃俱及泉。其井深若干？五等繩各長若干？

The solution presented in the *Guide to Calculation* then follows. First, the depth of the well is found, in the same manner that is recorded in earlier Chinese sources, including Jia's *Detailed Notes*, Yang's *Detailed Explanations*, and Wu's *Complete Collection of Mathematical Methods*, by a determinantal calculation:

> Lay out the five columns. Take the numbers of the five ropes as the "major terms" [*mu*]. Take 1, [the number of] ropes borrowed, as the "minor term" [*zi*]. First take 2 [representing the number of ropes from] A multiplied by the 3 [for the ropes from] B to obtain 6. Multiply by [4 for the ropes from] C to obtain 24. Multiply by [5 for the ropes from] D to obtain 120. Multiply by [6 for the ropes from] E to obtain 720. Add to this the "minor term" 1, and together, 721 is the "depth product" of the well.
>
> 列五行。以五繩之數為母。借繩一為子。先取甲二乘乙三得六。以乘丙得二十四。以乘丁得一百二十。以乘戊得七百二十。併入子一。共七百二十一為井深積。

The *Guide to Calculation* presents this calculation and the assignment of the result to the depth of the well without any commentary, analysis, or criticism, just as in Wu's *Complete Collection of Mathematical Methods*.

Following the determination of the depth of the well, the problem is set as an array:

> Arrange in position [in an array, as shown in Figure 2 on the next page].
>
> 列位。

五	四	三	二	一
甲一	○	○	○	甲二
○	○	○	乙三	乙一
○	○	丙四	丙一	○
○	丁五	丁一	○	○
戊六	戊一	○	○	○
七百二十一	七百二十一	七百二十一	七百二十一	七百二十一

5	4	3	2	1
A 1	0	0	0	*A* 2
0	0	0	*B* 3	*B* 1
0	0	*C* 4	*C* 1	0
0	*D* 4	*D* 1	0	0
E 5	*E* 1	0	0	0
721	721	721	721	721

Figure 2: Diagram from Li and Ricci's *Guide to Calculation*.

The elimination of entries by row reductions then follows:

> Then take the fifth column as "principal," and multiply [it] together with the first, second, third, and fourth [columns].
>
> 乃取五行為主，而以一二三四俱與相乘。
>
> First take 2, [the entry for] A in the first column, as the "divisor," and multiply the fifth column term-by-term. [The entry for] A, 1, gets 2. [The entry for] E, 6, gets 12. The constant term, 721, gets 1442.
>
> 先以一行甲二為法，遍乘五行 甲一得二，戊六得十二，積七百二十一得一千四百四十二。
>
> Also, multiply the first column term-by-term by 1, [the entry for] A in the fifth column, and subtract [the first column from the fifth] term-by-term. [The entry for] A, 2, has 2 subtracted from it and is eliminated. [The entry for] B, 1, gets 1, and because [the entry for] B in the fifth column is empty, set −1. [The entry for] the constant term, 721, gets the original number and is subtracted from the fifth column, then the remainder is still 721.
>
> 五行甲一亦乘一行對減。甲二得二減盡。乙一得一。因五行乙空立負一。積七百二十一得本數，以減五行，仍餘七百二十一。
>
> Next take 3, [the entry for] B in the second column, as the "divisor," and multiply the fifth column. [The entry for] B, −1, gets −3. [The entry for] E, +12, gets 36. [The entry for] the constant term, 721, gets 2163.
>
> 次以二行乙三為法乘五行。乙負一得負三。戊正十二得三十六。積七百二十一得二千一百六十三。

Also multiply the second column by −1, [the entry for] B in the fifth column. [The entry] B, 3, has 3 subtracted from it and is eliminated. [The entry for] C, 1, gets 1, and because [the entry for] C in the fifth column is empty, set −1. [The entry for] the constant term, 721, gets the original number, [to which is] added the constant term of the fifth column, 2163, together 2884.

五行乙負一亦乘二行。乙三得三，對減盡。丙一得一，因五行丙空，立負一。積七百二十一得本數併入五行積二千一百六十三，共二千八百八十四。

Next take 4, [the entry for] C in the third column, as the "divisor," and multiply the fifth column. [The entry for] E, +36, gets 144. [The entry for] the constant term, 2884, gets 11536.

再以三行丙四為法，乘五行。戊正三十六得一百四十四。積二千八百八十四得一萬一千五百三十六。

Also multiply the third column by −1, [the entry for] C in the fifth column. [The entry for] B, 4, has 4 subtracted from it and is eliminated. [The entry for] D, 1, gets 1. Because [the entry for] D in the fifth column is empty, set −1. [The entry for] the constant term gets the original number, and subtracted from [the entry in] the fifth column, the remainder is 10815.

五行丙負一亦乘三行。丙四得四減盡。丁一得一，因五行丁空，立負一。積得本數，與五行積一萬一千五百三十六對減餘一萬八百一十五。

Next take 5, [the entry for] D in the fourth column, as the "divisor," and multiply the fifth column. [The entry for] D, −1, obtains 5. [The entry for] E, +144, obtains 720. The constant term, 10815, obtains 54075.

又以四行丁五為法乘五行。丁負一得五。戊正一百四十四得七百二十。積一萬八百一十五得五萬四千七十五。

Also multiply the fourth column by −1, the entry for D in the fifth column. [The entry for] D, 5, obtains 5 and is eliminated. [The entry for] E, 1, obtains 1, added to 720, [the entry for] E in the fifth column, together is 721. [The entry for] the constant term obtains the original number, added to [the entry for] the constant term in the fifth column, 54075, obtaining 54796.

五行丁負一亦乘四行。丁五得五減盡。戊一得一，併入五行戊正七百二十，共七百二十一。積得本數併入五行積五萬四千七十五，得五萬四千七百九十六。

Then use the values obtained from the last [calculation] to find it [the solution]. Take the constant term 54796 as the dividend, and 721 for E as the divisor, divide it, obtaining 7 *chi* 6 *cun* for E's rope. Subtract [that] from the total constant term in the fourth column. 721. The remainder is 645, and divide it by 5, [the entry for] D, obtaining 129, giving 1 *zhang* 2 *chi* 9 *cun* for D's rope. Subtract [that] from the constant term in the third column. 721, the same for the following. The remainder is 592, and divide it by 4, [the entry for] C, obtaining 1 *zhang* 4 *chi* 8 *cun* for C's rope. Subtract [that] from the constant term in the second column, the remainder is 573, and divide by 3, [the entry for] B, obtaining 1 *zhang* 9 *chi* 1 *cun* for B's rope. Subtract [that] from the constant term in the first column, the remainder is 530, and divide it by 2, [the entry for] A, obtaining 2 *zhang* 6 *chi* 5 *cun* for A's rope.

乃以最後所得求之。以積五萬四千七百九十六為實，戊七百二十一為法，除之，得戊繩七尺六寸。以減四行總積，七百二十一餘六百四十五，以丁五除之，得一百二十九，為丁繩一丈二尺九寸。以減三行積，七百二十一，後同餘五百九十二，以丙四除之，得丙繩一丈四尺八寸。亦減二行積，餘五百七十三，以乙三除，得乙繩一丈九尺一寸。以減一行積，餘五百三十，以甲二除，得甲繩二丈六尺五寸。 (*TWSZ*, *Tong bian*, *juan* 5, 18a–19b)

The version of the "well problem" in the *Guide to Calculation* contains no significant additions or improvements, and is presented without additional analysis or criticisms. Most of the differences are quite minor and terminological: the fifth column is called "principal" (*zhu* 主) apparently because it is the only column that is transformed; the pivots are called "divisors" (*fa* 法); the term "set negative" (*li fu* 立負) is used for entering negative numbers. Because we do not know which mathematical treatise they used as their source, we cannot know if any of this terminology is original, but most likely it is not. For example, pivots are called "divisors" (*fa*) in Wu's *Complete Collection of Mathematical Methods*; the phrase "set negative nine" (*li fu jiu* 立負九) appears in Cheng's *Comprehensive Source of Mathematical Methods*. There is no evidence that the compilers of the *Guide to Calculation* understood these methods for solving *fangcheng*. The silence of the *Guide to Calculation* stands in stark contrast to Mei Wending's "On *Fangcheng*," written just half a century later, in which Mei offers 40 pages of criticism of the "well problem" alone.[51] This suggests that the compilers of the *Guide to Calculation* not only did not understand the mathematics here, they did not even notice that there was anything unusual about the method for calculating the depth of the well.

In addition to the well problem, there are three problems copied into the *Guide to Calculation* that are of the form of augmented matrices (8) and (14). These problems are given in Table 4.

Table 4: From among the nineteen *fangcheng* problems copied into the *Guide to Calculation*, below are three problems that are variants of augmented matrices (8) and (14), in addition to the well problem, which is problem 19 in the *Guide to Calculation*.

Problem	*Chinese*	*Augmented Matrix*	*Notes*
14	借牛一 ○ 正驢三 七百斤 ○ 正馬二 借驢一 七百斤 正牛一 借馬一 ○ 七百斤	$\begin{bmatrix} 1 & -1 & 0 & 700 \\ 0 & 2 & -1 & 700 \\ -1 & 0 & 3 & 700 \end{bmatrix}$	

Continued on the next page

[51] Mei Wending 梅文鼎 (1633–1721), *Fangcheng lun* 方程論 [On *fangcheng*], in *ZKJDT*, *juan* 4, 40a–60a.

Problem	*Chinese*	*Augmented Matrix*	*Notes*
15	硃二 黄三 ○ 價三千四十文 ○ 黄五 碌六 價六百四十文 硃三 ○ 碌七 價二千九百八十文	$\begin{bmatrix} 2 & 3 & 0 & 2040 \\ 0 & 5 & 6 & 640 \\ 3 & 0 & 7 & 2980 \end{bmatrix}$	Similar to problem 5 from Cheng's *Comprehensive Source of Mathematical Methods*.
18	甲柰二 梨四 ○ ○ 四十文 乙○ 梨二 桃七 ○ 四十文 丙○ ○ 桃四 榴七 三十文 丁柰一 ○ ○ 榴八 二十四文	$\begin{bmatrix} 2 & 4 & 0 & 0 & 40 \\ 0 & 2 & 7 & 0 & 40 \\ 0 & 0 & 4 & 7 & 30 \\ 1 & 0 & 0 & 8 & 24 \end{bmatrix}$	Similar to problem 8 from Cheng's *Comprehensive Source of Mathematical Methods*.

The Ten Classics of Chinese Mathematics as "Tattered Sandals"

We are now in a better position to evaluate Xu Guangqi's "Preface at the Printing of the *Guide to Calculation in the Unified Script*" (*Ke Tong wen suan zhi xu* 刻同文算指序),[52] dated spring of 1614 (*Wanli jia yin chun yue* 萬曆甲寅春月), which illustrates the strategies he employed to assert the superiority of "Western Learning":

> The origin of numbers, could it not be at the beginning of human history? Starting with one, ending with ten, the ten fingers symbolize them and are bent to calculate them, [numbers] are of unsurpassed utility! Across the five directions and myriad countries, changes in customs are multitudinous. When it comes to calculating numbers, there are none that are not the same; that all possess ten fingers, there are none that are not the same.
>
> 數之原其與生人俱來乎？始於一，終於十，十指象之。屈而計諸，不可勝用也。五方萬國，風習千變。至于算數，無弗同者，十指之賅存，無弗同耳。

[52] Xu Guangqi, *Ke Tong wen suan zhi xu* 刻同文算指序 [Preface at the printing of the Guide to calculation in the unified script], in *ZKJDT*.

In China, [beginning] from [the time] that the Yellow Emperor ordered Li Shou to do calculations in order to help Rong Cheng,[53] [it] was during the Zhou Dynasty (1045?–256 BCE) that [mathematics] reached its great completeness. The Duke of Zhou used it [mathematics], giving it a place in the curriculum used to choose officials, to promote the capable and virtuous [to the Imperial College],[54] to appoint them to be officials. Among the disciples of Confucius, those who mastered the Six Arts[55] were praised as having "ascended the hall and entered the chamber."[56] If mathematics were to fall to waste, the teachings of the Duke of Zhou and Confucius would fall into disorder. Some state that records and books were burned by Mr. Ying [the first emperor of China], and most of the learning of the Three Dynasties [Xia, Shang, and Zhou (ca. 2100?–256 BCE)] was not transmitted. If so, for all the early Confucians of the time of Ma [Rong (79–166)][57] and Zheng [Xuan (127–200)],[58] what was left for them to transmit? Of the *Ten Classics* [of mathematics][59] listed in

[53] Rong Cheng was the (perhaps mythical) great minister who aided the Yellow Emperor in establishing the calendar.

[54] This phrase is used in the preface to the *Nine Chapters*; it is also copied in Wu Jing's *Nine Chapters, Methods Arranged by Categories* (*Jiu zhang bi lei fa* 九章比類法, ca. 1450).

[55] The phrase *shen tong liu yi* appears in a description of Confucius' disciples in the *Records of the Grand Historian*: "Confucius taught [the classics of] Poetry, Documents, Rites, and Music, with approximately 3000 followers, those who were masters of the Six Arts [numbered] seventy-two 孔子以詩書禮樂教，弟子蓋三千焉，身通六藝者七十有二人. Sima Qian 司馬遷 (c. 145–c. 86 BCE), *Shiji ji jie* 史記集解 [Records of the grand historian, with collected explanations], in *SKQS*.

[56] The phrase *sheng tang ru shi* 升堂入室 originates from a passage in the *Analects*: "The disciples were not respectful toward Zilu. Confucius stated, 'You [Zilu], he has ascended the hall, but has not entered the chamber'" 門人不敬子路。子曰：由也升堂矣，未入於室也, using Zilu's shortcomings to indicate different levels of attainment. Later this phrase was used to indicate profound attainment. I thank Jongtae Lim for his many helpful suggestions and corrections on this and the following sentences.

[57] Ma Rong 馬融 (79–166) of the Han dynasty wrote annotations for the *Classic of Poetry*, *Classic of Changes*, *Classic of Rites*, *Classic of Documents*, *Classic of Filiality*, *Analects*, *Laozi*, *Huainanzi*, and others.

[58] Zheng Xuan 鄭玄 (127–200) of the Han dynasty studied the classics, including the *Gongyang Commentary* and the *Zuo Commentary* on the *Spring and Autumn Annals*, the *Record of Rites*, and the ancient text version of the *Classic of Documents*. He also studied the astronomical treatise *San tong li* 三統曆 and the *Nine Chapters of the Mathematical Arts*. His extant works include commentaries on the Mao commentary on the *Classic of Poetry*, the *Rites of Zhou*, *Record of Rites*, and *Decorum and Rites*; his commentaries on the *Classic of Changes*, *Spring and Autumn Annals*, and several other works were lost and exist only as reconstructed fragments.

[59] The term *Ten Classics* is also used to refer to the Confucian classics. Here Xu is referring to the ten classics of mathematics. Cheng, *Suanfa tongzong*, lists the *Ten Classics* as follows: 黃帝九章, 周髀算經, 五經算法, 海島算經, 孫子算法, 張建算法, 五曹算法, 緝古算法, 夏侯算法, 算術拾遺.

the *Six Canons of the Tang*,[60] what books were left for the Erudites and disciples who studied for five years?

我中夏自黃帝命隸首作算，以佐容成，至周大備。周公用之，列於學官以取士，賓興賢能，而官使之。孔門弟子身通六藝者，謂之升堂入室，使數學可廢，則周孔之教踳矣。而或謂載籍燔於嬴氏，三代之學多不傳，則馬、鄭諸儒先，相授何物？唐六典所列十經博士弟子，五年而學成者，又何書也？

From the above, it can be stated only that the learning of calculation and numbers has especially decayed over the most recent several hundreds of years. There are two reasons for this decay: one is Confucians who dispute [unimportant] principles and despise all evidential matters under heaven; one is the absurd techniques that state that numbers have spiritual principles, capable of knowing what is to come and preserving what has gone, there is nothing they cannot do. In the end, among the spiritual there is not one that is effective; and among the evidenced there is not one that is preserved. The great methods used by the sages from the past to order the world and benefit [the people], once the literati were not able to obtain them, completely following the arts and tasks of governmental affairs of antiquity [Xia, Shang, and Zhou dynasties] became a distant [ideal].

由是言之，算數之學，特廢於近世數百年間爾。廢之緣有二：其一為名理之儒，士苴天下之實事；其一為妖妄之術，謬言數有神理，能知來藏往，靡所不效。卒於神者無一效，而實者亡一存。往昔聖人所以制世利用之大法，曾不能得之士大夫間，而術業政事，盡遜於古初遠矣。

My friend Li Zhenzhi [Zhizao] of the Bureau of Waterways and Irrigation, a highly esteemed man of understanding, has, together with me, long despaired because of this state [of the decline of mathematics]. Thus [we] sought contemporary books on the techniques of calculation. [In these] probably only one-tenth is writings from the beginning of antiquity, eight-tenths is vulgar transmissions of recent writers, and another one-tenth is transmissions by the early Confucians that do not betray [the works of] the beginning of antiquity, that is all. I have once cursorily examined[61] these vulgar transmissions, which are techniques of recluses, and most are specious absurdities not [meriting] discussion. Even in the writings allegedly from the beginning of antiquity and those that [allegedly] do not betray the beginning of antiquity [of the early

[60] A text describing government laws, regulations, and institutions in the Tang dynasty (618–906), traditionally held to have been written during the reign of the emperor Xuan Zhong (712–756). It contains a passage describing the Mathematics School, to which Xu is referring. For a description of the Mathematics School, see Charles O. Hucker, *A Dictionary of Official Titles in Imperial China* (Stanford, CA: Stanford University Press, 1985), s.v.

[61] A tentative translation for *xi mu* 戲目.

Confucians], there is nothing more than just the methods, without being able to state the intent behind establishing the methods. Moreover, again distantly thinking of the study of the *Ten Classics* [of mathematics] in the Tang dynasty, there must have been original, ultimately complete, and profoundly subtle meanings. If they stopped at the contemporary transmissions, then they could be mastered in two months—what work could require five years?

余友李水部振之，卓犖通人，生平相與慨嘆此事，行求當世算術之書，大都古初之文十一，近代俗傳之言十八，其儒先所述作，而不倍於古初者，亦復十一而已。俗傳者，余嘗戲目為閉關之術，多謬妄弗論，即所謂古初之文，與其弗倍於古初者，亦僅僅具有其法，而不能言其立法之意。益復遠想，唐學十經，必有原始通極微渺之義。若止如今世所傳，則浹月可盡，何事乃須五年也？

Now that [I] together with [Li] accompany Mr. Ricci of the Western countries, in our spare time while discussing the Way, [we] have often touched on principles and numbers. Since [his] discussions of the Way (*dao*) and Principle (*li*) all return to basics and are solid (*shi*), they absolutely dispel all theories of emptiness, profundity, illusion, and absurdity; and the studies of the numerical arts can all be traced back to the origins to recover [the proper] tradition, the root supporting the leaves and branches, above exhausting the nine heavens, on [all] sides completing the myriad affairs. In the Western Countries in the academies of antiquity,[62] it also took several years to complete the studies [of mathematics]. Even though our generation cannot see the *Ten Classics* of the Tang Dynasty, looking at the calendar and all the affairs talked about by Mr. Ricci together with all the teachers of the same aspirations [the Jesuits],[63] their mathematics is precise and subtle, ten or one hundred times that compared with [the mathematics] of the Han and Tang dynasties. Because of this [we] took seats and asked to be benefited [by the teachings of Ricci]. Unfortunately, because of our comings and goings, Zhenzhi and I missed each other.

既又相與從西國利先生游，論道之隙，時時及於理數。其言道言理，既皆返本蹠實，絕去一切虛玄幻妄之說；而象數之學，亦皆溯[64]源承流，根附葉著，上窮九天，旁該萬事，在於西國膠庠之中，亦數年而學成者也。吾輩既不及睹[65]唐之十經，觀利公與同志諸先生所言曆法諸事，即其數學精妙，比於漢唐之世，十百倍之，因而造席請益。惜余與振之出入相左。

[62] Literally, the term *jiao xiang* 膠庠 refers to academies of the Zhou dynasty; presumably Xu is likening antiquity in Greece to the Zhou dynasty.

[63] Xu uses this circumlocution probably to avoid having to explicitly translate the "Society of Jesus."

[64] Substituted for a variant form of the character.

[65] Substituted for variant form.

Zhenzhi came twice to live in Beijing and translated several chapters of Ricci's mathematics. Since it was already a manuscript, I began to inquire and [we] read it together, and discussed it together. Overall, those [Western mathematical techniques] that were the same as the old (*jiu*)[66] [Chinese] techniques were ones in which the old [Chinese techniques] did not reach [the Western techniques]; those [Western mathematical techniques] that differed from the old [Chinese] techniques were ones that the old [Chinese techniques] did not have. Taking the old [Chinese] techniques, [we] read them together and discussed them together. Overall, of those [Chinese techniques] that were compatible with the Western techniques, there were none that were not compatible with principle (*li*); of those [Chinese techniques] that were mistaken according to the Western ones, there were none that were not mistaken according to principle (*li*).[67] Because of this, Zhenzhi took the old techniques and considered them, discarding and selecting, using the translated Western techniques and appending [them] in parallel, printed [the manuscript], and named it *Guide to Calculation in the Unified Script*.[68] This can be called encompassing the beauty of the arts and studies, opening the path for [further] writing. Although the *Ten Classics* [of mathematics] are lost, it is just like discarding tattered sandals.[69] …

振之兩度居燕，譯得其算術如[70]干卷，既脫稿，余始間請而共讀之，共講之，大率與舊術同者，舊所弗及也；與舊術異者，則舊所未之有也。旋取舊術而共讀之，共講之，大率與西術合者，靡弗與理合也；與西術謬者，靡弗與理謬也。振之因取舊術，斟酌去取，用所譯西術駢附，梓之，題曰：同文算指，斯可謂網羅藝業之美，開廓著述之途，雖失十經，如棄敝屩矣。…

The assertions by modern historians that mathematics during the Ming dynasty (1368–1644 CE) had "fallen into oblivion" have often been supported by little more than Xu's claim from his preface that "the learning of calculation and numbers has especially decayed over the most recent several hundreds of years." Modern historians (both Chinese and Western) have asserted that Chinese mathematics reached its zenith during the Song Dynasty (960–1279 CE), but Xu is asserting that Chinese mathematics reached its zenith about two thousand years before his lifetime, during the Zhou Dynasty (1045?–256 BCE). Xu dismisses the entire Chinese mathematical tradition: the few works remaining from antiquity are unworthy of antiquity, the early Confucians' works are corrupt, and contemporary works are vulgar. Western Learn-

66 The term that Xu uses here for the Chinese is "old" (*jiu* 舊) in the pejorative sense, in contrast to the term "beginning of antiquity" (*gu chu* 古初), used in a very positive sense.

67 These two sets of sentences are written in parallel form, similar to an examination essay.

68 Again, the term *tongwen* 同文 appears in the *Classic of Rites*: 車同軌，書同文. Li Zhizao seems to use it in the sense of translated into the same [Chinese] script.

69 Literally, *jue* 屩 means cloth or straw shoes, *bi* 敝 means decrepit, shabby, dilapidated.

70 *Ru* 如 (Wang Zhongmin's edition) should be *ruo* 若.

ing, Xu claims, is in every case superior to the Chinese. He offers no specifics about mathematical techniques; his claims are not based on any knowledge or evaluation of Chinese mathematics.

Despite Xu's appeals to antiquity, we should not mistake him for a textualist interested in studying early texts. There is little evidence that Xu inquired into Chinese mathematical treatises, compared with the work of his contemporaries. The itinerant merchant Cheng Dawei includes an extensive bibliography in his *Comprehensive Source of Mathematical Methods*. Jiao Hong had written a comprehensive bibliography, the *Guo shi jing ji zhi* 國史經籍志;[71] although Xu mentions this work in a preface he later wrote for Jiao Hong, he never mentions its extensive record of mathematical treatises. Seven years after the translation of Euclid's *Elements*, Xu remains unfamiliar with even the titles of all but the most well-known Chinese mathematical texts. Furthermore, contrary to Xu's claims, the *Ten Classics* of Chinese mathematics were not lost during the Ming dynasty: many were readily available;[72] all were extant and included in the *Great Compendium of Yongle* (*Yongle dadian*).[73] Not only does Xu know very little about Chinese mathematics, he evinces little interest in Chinese texts themselves.

Conclusions

The evidence in the *Guide to Calculation in the Unified Script* gives us more insight into the propaganda promoting Western Learning. We should not mistake Xu's claims of the decline of Chinese mathematics or his dismissal of the entire Chinese tradition as beliefs that he actually held—this was simply propaganda that he wrote to promote Western Learning. Though Xu Guangqi and his collaborators did not understand much about Chinese mathematics, they did understand how advanced parts of it were, to the extent that the most difficult problems they included in their *Guide to Calculation* were copied directly from the sources they denounced as vulgar.

This parallels their borrowings from Buddhism, which they similarly denounced. In the same fashion as they had with Buddhism, they borrowed and appropriated from Chinese mathematical texts. They then transformed their copy into the original, by claiming that their copy predated the Chinese original and thus recovered meanings lost in antiquity; at the same time, they vehemently attacked the Chinese original as a corruption of their copy. We should not, of course, believe their propaganda. This article demonstrates, I think, that they did not believe it themselves: as they uncomprehendingly but carefully copied from Chinese mathematical texts that they

71 Jiao Hong 焦竑 (1541–1620), *Guo shi jing ji zhi* 國史經籍志 [Record of books for the dynastic history] (Taibei: Taiwan shangwu yinshuguan 臺灣商務印書館, 1965).

72 For a study of extant bibliographies and an analysis of the mathematical treatises available during the Ming dynasty, see Hart, *Chinese Roots of Linear Algebra*, Appendix B, "Chinese Mathematical Treatises," pp. 213–54.

73 This is the source from which Dai Zhen and the Qing compilers of the *Siku quan shu* "discovered" these treatises.

dismissed as vulgar, they could not have believed that they were actually recovering lost meanings from the ancients.

Circulation of Linear Algebra across Eurasia

The broader purpose of this article is to question the assumption that the advent of the Jesuits in late Ming China marked the introduction of Western science into China and the "first encounter" of China and the West. To do so, we have traced the history of a distinctive class of systems of linear equations with n conditions in $n+1$ unknowns, exemplified by augmented matrix (8), recorded in the *Nine Chapters* in about the first century CE. Evidence preserved in the *Nine Chapters* and in later commentaries demonstrates the following: (1) Determinantal calculations were used to set the value of the $(n+1)^{\text{th}}$ unknown; although the earliest extant Chinese record of such a determinantal calculation is a commentary dating from the twelfth century, it is likely that these calculations were used at the time of the compilation of the *Nine Chapters* in about 100 CE. (2) Determinantal solutions for all the unknowns were also known; although the first extant record of determinantal solutions is a commentary from the seventeenth century, evidence suggests that some determinantal solutions were known at the time of the compilation of the *Nine Chapters*. Essentially identical determinantal calculations and solutions are recorded in the *Liber Abaci* (1202) of Leonardo Pisano (Fibonacci). Although his problems were not displayed on a counting board, the terms were displayed as a series of fractions arranged contiguously, permitting similar calculations. First, the value of the $(n+1)^{\text{th}}$ unknown was assigned by calculating what we would in modern terms call the numerator of the determinant of the coefficient matrix. Subsequently, determinantal-style calculations resulted in the values of the remaining n unknowns.

Toward a World History of Science

This article, having demonstrated the diffusion of these problems, marks only the beginning of a broader investigation of the circulation of mathematical practices across the Eurasian continent prior to the West's "scientific revolution." In China, these mathematical practices were what we might call non-scholarly, neither based on nor transmitted primarily by texts. That is, practices such as linear algebra (*fangcheng*) were the specialty of anonymous and likely illiterate adepts; their practices were only occasionally recorded by literati who compiled treatises on the mathematical arts in their pursuit of imperial patronage; although these literati understood the basics, they did not understand (and in fact expressed disdain toward) the more esoteric calculations. What extant Chinese texts preserve is fragmentary evidence from which we can only attempt to reconstruct these practices.[74] The discoveries we now attribute to Leonardo (Fibonacci) may themselves have been the product of

[74] Hart, *Chinese Roots of Linear Algebra*.

non-scholarly traditions.[75] The diffusion of these practices across Eurasia was likely effected by merchants, missionaries, and travelers, following the routes by which commerce, art, and religion circulated in what is increasingly understood to be the "global Middle Ages," to use Geraldine Heng's helpful term. This hypothesis may help explain the otherwise seemingly inexplicable appearance of Gaussian elimination, identical to Chinese methods, in the work of the French monk Jean Borrel in the sixteenth century. This may also help explain the almost simultaneous appearance in the seventeenth century of determinants in the works of Leibniz, a Sinophile, and the Japanese mathematician Seki, who had studied Chinese mathematics. The considerable variety of problems, preserved in extant Chinese treatises, suggests that determinants were not suddenly discovered *ex nihilo*, but emerged from hundreds of years of explorations of increasingly complex patterns of determinantal-style computations.[76]

More broadly, the evidence presented in this article suggests the need to rethink the world history of science prior to the scientific revolution. The fact that problems this specialized, with solutions this esoteric, eventually spread across Eurasia—including early imperial China and thirteenth-century Italy—suggests that the assumption that other mathematical and scientific practices were not similarly transmitted and diffused should be reconsidered. If we are to pursue such efforts, we must reconsider the relationships between scientific practices, texts, and authorship during this period. The scientific practices of the period often did not depend on texts: indeed, the learning, teaching, and transmission of these practices did not require literacy; and when these practices were recorded in texts, it was commonly for purposes of patronage or, less frequently, displays of expertise. That is, we must take care to distinguish clearly between the historical archive—in this case, scientific writings that have been fortuitously preserved—and the world of scientific practice. Given this concern, it makes little sense for historians to obligingly grant credit for scientific discoveries to those who, in their pursuit of patronage, sought to claim that credit for themselves. It makes even less sense to attribute credit to what we now anachronistically call "China" or "the West," relying merely on the earliest known extant text in which a practice is recorded. The assumption that specific scientific practices belong to "China," "Islam," or "the West" is just that, an assumption, and one that resulted in part from the twentieth-century focus on civilizations and their comparison in the history of science. It is more likely that, like the linear algebra problems documented in this article, scientific practices had centers of activity that shifted over time to and between different parts of Eurasia. In the twenty-first century, it should be the task of the world history of science to trace these practices and their global circulation.

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[75] Jens Høyrup, "Leonardo Fibonacci and Abbaco Culture: A Proposal to Invert the Roles," *Revue d'histoire des mathèmatiques* 11 (2005): 23–56.

[76] Jean-Claude Martzloff, in his review of *Chinese Roots of Linear Algebra*, suggests an important possible direction for future research: "in the context of Japanese traditional mathematics, the notion of determinant developed by Seki Takakazu was prompted not exactly by the study of linear systems but rather by the problem of the elimination of unknowns between non-linear polynomial systems of equations (and the same is true in the case of Leibniz)." See Jean-Claude Martzloff, "Review of *The Chinese Roots of Linear Algebra*, by Roger Hart," *Zentralblatt MATH* (2011).

Abstract

Xu Guangqi (1562–1633), Li Zhizao (1565–1630), and Yang Tingyun (1557–1627) — together known as the "Three Pillars" of Catholicism in late Ming China — promoted the Western mathematics brought to China by the Jesuits as superior to contemporary Chinese mathematics. Xu, the most prominent among them, in a preface for their purported "translation" of a European mathematical treatise, denounced Chinese mathematics as "tattered sandals" to be "discarded," because Western mathematics was, he claimed, in every way superior. I argue that they did not themselves believe these claims, for they purloined what is arguably the most difficult mathematics in their "translation"—*fangcheng*, or what we would now call linear algebra—from the very Chinese mathematical treatises Xu execrated. I trace the mathematical practices behind these *fangcheng* problems, reconstructing these practices as solved on two-dimensional counting boards. I show how adepts were able to compute solutions to difficult problems involving *n* conditions and *n* unknowns with only counting rods and an understanding of simple two-dimensional patterns. I present evidence that specialized *fangcheng* problems with solutions so arcane that they can serve as "fingerprints" circulated across the Eurasian continent, and can be found in Italian texts from the thirteenth century. The advent of the Jesuits in China was thus hardly the "first encounter" of China and the West. Instead, world history of science must trace the global circulation of scientific practices.

Key Words: world history of science, history of linear algebra, history of Chinese mathematics, *fangcheng*, Xu Guangqi, Matteo Ricci, Fibonacci